# Finite expansions

Definition: Let $f$ be a real function defined on an interval $I$ and let $a \in I$ and $n \in \mathbb{N}$. We say that $f$ has a finite expansion to order $n$ near $a$ if there exist $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and a function $\varepsilon(x)$ such that

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n \varepsilon(x)$$

when $\varepsilon(x) \longrightarrow 0$ when $x$ tends to $a$.

Let $P_n(x-a) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$ and

$R_n(x) = (x-a)^n \varepsilon(x)$ so $f(x) = P_n(x-a) + R_n(x)$.

$P_n(x-a)$: regular part of the finite expansion of $f$ near $a$ to order $n$.

$R_n(x) = (x-a)^n \varepsilon(x)$: remainder part of the finite expansion of $f$ near $a$ to order $n$.

* we denote by $f.\,C_n(a)$ to be the finite expansion of $f$ near $a$ to order $n$

particular case: $a=0$

$$f(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n+x^n\varepsilon(x) \,/\, \varepsilon(x)\xrightarrow[x\to 0]{} 0.$$

$f(x)\underset{0}{\sim} P(x)=a_0+a_1x+a_2x^2+\dots+a_nx^n$: we approximate the function $f$ near $0$ by a polynomial $P_n(x)$.

Remark: 1- the change of variable "$t=x-a$" permits to study the finite expansion of the function $g(t)=f(a+t)$ close to $0$ instead of studing the f.e of $f(x)$ close to $a$, that is:

$t=x-a \Leftrightarrow x=a+t$

$x\to a \Leftrightarrow t\to 0.$

$f(x)=f(a+t)=g(t).$

$$g(t)=a_0+a_1t+a_2t^2+\dots+a_nt^n+t^n\varepsilon(t) \,|\, \varepsilon(t)\xrightarrow[t\to 0]{} 0$$

$$\Rightarrow f(x)=g(x-a)=a_0+a_1(x-a)+a_2(x-a)^2+\dots+a_n(x-a)^n+(x-a)^n\varepsilon(x-a)$$

where $\varepsilon(x-a)\xrightarrow[x\to a]{} 0.$

2- The change of variable $t=\frac{1}{x}$ allows us to study the function $g(t)=f(\frac{1}{t})$ close to $0$ instead of study the function $f(x)$ close to $\pm\infty$. ($t=\frac{1}{x}\to 0 \Leftrightarrow x\to\infty$).

## Finite expansion near 0

**Theorem (Taylor Formula)**: Let $f$ be a function of class $C^n$ on $I$ and let $0 \in I$, then $f$ has $f.e_n(0)$ given by:

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x)$$

where $\varepsilon(x) \underset{x \to 0}{\longrightarrow} 0$

Let $a_0 = f(0)$, $a_1 = \frac{f^{(1)}(0)}{1!}$, $\dots$, $a_k = \frac{f^{(k)}(0)}{k!}$ $k = 0, \dots, n$

$$\Rightarrow f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n\varepsilon(x).$$

Example

1) $f(x) = e^x$, $a = 0$

$f^{(k)}(x) = e^x \ \forall k \in \mathbb{N} \Rightarrow f^{(k)}(0) = e^0 = 1$

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x)$$

$$\Rightarrow e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n + x^n\varepsilon(x)$$

$$\Rightarrow \boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + x^n\varepsilon(x)}$$

2) $f(x) = \ln(1+x)$ , $a = 0$

$$f^{(k)}(x) = \frac{(-1)^{k-1}\cdot(k-1)!}{(1+x)^k} \;,\; k \in \mathbb{N}^* \Rightarrow f^{(k)}(0) = \frac{(-1)^{k-1}(k-1)!}{1}$$

$$= (-1)^{k-1}(k-1)!$$

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x)$$

$$\Rightarrow \ln(1+x) = 0 + \frac{1}{1!}x - \frac{1}{2}x^2 + \frac{2!}{3!}x^3 + \dots + \frac{(-1)^n(n-1)!}{n!}x^n + x^n\varepsilon(x)$$

$$\Rightarrow \boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^n}{n}x^n + x^n\varepsilon(x)}.$$

Finite expansions near 0 of some usual functions

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + x^n\varepsilon(x)}$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + x^n\varepsilon(x)}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\varepsilon(x)}$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n}\varepsilon(x)}$$

$$\boxed{(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 \dots + \frac{\alpha(\alpha-1)\dots(\alpha-(n-1))}{n!}x^n + x^n\varepsilon(x)}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + x^n \varepsilon(x).$$

$$\boxed{ch\,x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + x^{2n}\varepsilon(x)}$$

$$\boxed{sh\,x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\varepsilon(x)}$$

Example: F.e$_3$(0) of $\sqrt{1+x}$

$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = (1+x)^{\alpha}$ where $\alpha = \frac{1}{2}$ (independ of $x$)

$$(1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha.(\alpha-1)}{2!}x^2 + \frac{\alpha.(\alpha-1)(\alpha-2)}{3!}x^3 + x^3\varepsilon(x)$$

$$\Rightarrow (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}x^3 + x^3\varepsilon(x)$$

$$\Rightarrow (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x)$$

propaties:

1- the finite expansion, if it exists, is unique

2. the principal part (regular part) of the finite expansion of an even (respectively, odd) function near 0 is an even (resp, odd) polynomial, i.e, it contains the even (resp, odd) powers of $x$ only

3. (Truncation): If $P_n(x) = \sum_{k=0}^{n} a_k x^k$ is the principal part to order $n$ at $0$ of $f$, then for every positive integer $m$ belonging to $[[0, n]]$, $\sum_{k=0}^{m} a_k x^k$ is the principal part to order $m$ of $f$ at $0$.

if $f(x) = \sum_{k=0}^{n} a_k x^k + x^n \varepsilon(x)$ then for $m \leq n$, $f(x) = \sum_{k=0}^{m} a_k x^k + x^m \varepsilon(x)$

proposition (Limit and continuity): suppose $f$ has $f.e_n(0)$, $n \geq 0$. Let $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be it's regular part to order $n$, then $f$ has $a_0$ as limit at $0$.

proposition (Differentiability): suppose that $f$ has $f.e_n(0)$ where $n \geq 1$. Let $P_n(x) = \sum_{k=0}^{n} a_k x^k$ be it's principal part to order $n$ and $g$ the extension by continuity of $f$ at $0$. Then $g$ is differentiable at $0$ and $g'(0) = a_1$.

proof: $g(x) = g(0) + a_1 x + x\varepsilon(x)$

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} \frac{a_1 x + x\varepsilon(x)}{x} = a_1.$$

## Operations on the finite expansions:

### 1. Sum

If $f$ and $g$ have finite expansions to order $n$ at $0$ where $P_n(x)$ and $q_n(x)$ are their regular parts to order $n$ then $f+g$ has a $f.e_n(0)$ and it's regular part to order $n$ is $P_n(x)+q_n(x)$. i.e

$$f(x)=P_n(x)+x^n\varepsilon_1(x) \mid \varepsilon_1(x)\xrightarrow[x\to 0]{} 0$$

$$g(x)=q_n(x)+x^n\varepsilon_2(x) \mid \varepsilon_2(x)\xrightarrow[x\to 0]{} 0$$

$$\Rightarrow f(x)+g(x)=P_n(x)+q_n(x)+x^n\underbrace{(\varepsilon_1(x)+\varepsilon_2(x))}_{\varepsilon(x)}$$

$$=P_n(x)+q_n(x)+x^n\varepsilon(x) \mid \varepsilon(x)\xrightarrow[x\to 0]{} 0$$

Example: $F.e_3(0)$ of $Ln(1+x)+\sin x$.

$$Ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+x^3\varepsilon(x)$$

$$\sin x=x-\frac{x^3}{3!}+x^3\varepsilon(x)$$

$$Ln(1+x)+\sin x=\left(x-\frac{x^2}{2}+\frac{x^3}{3}+x-\frac{x^3}{3!}\right)+x^3\varepsilon(x)$$

$$=2x-\frac{x^2}{2}+\frac{1}{6}x^3+x^3\varepsilon(x)$$

2. If $f$ has a finite expansion to order $n$ at $0$, where $p_n(x)$ be it's regular part then for all $\lambda \in \mathbb{R}$, $\lambda f$ has a finite expansion to order $n$ ~~and~~ near $0$ and it's regular part to order $n$ is $\lambda p_n(x)$

Example: Calculate $f.C_5(0)$ of $2\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\varepsilon(x)$$

$$2\cos x = 2 - x^2 + \frac{x^4}{12} + x^5\varepsilon(x).$$

3. product:

If $f$ and $g$ have finite expansions to order $n$ at $0$, where $p_n(x)$ and $q_n(x)$ are their regular parts to order $n$ then $f.g$ has a $f.e_n(0)$ and it's regular part to order $n$ is obtained by forming the product $p_n(x) \cdot q_n(x)$ and retaining only the terms of degree less than or equal to $n$.

Example: $F.e_3(0)$ of $\sin x . e^x$.

$$\sin x = x - \frac{x^3}{3!} + x^3\varepsilon(x)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3\varepsilon(x)$$

$$\sin x \cdot e^x = \left(x - \frac{x^3}{6}\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) + x^3 \varepsilon(x)$$

$$= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} + x^3 \varepsilon(x)$$

$$\boxed{\sin x \cdot e^x = x + x^2 + \frac{1}{3} x^3 + x^3 \varepsilon(x)}$$

Example: F.e$_5$(0) of $x \cdot \sin x$.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)$$

$$x \sin x = x^2 - \frac{x^4}{6} + x^5 \varepsilon(x).$$

Definition (Valuation of a function):

Given a non zero polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$, the Valuation of $p(x)$, denoted by $Val(p)$ is the smallest $k \in \mathbb{N}$ such that $a_k \neq 0$

The valuation of $f$ is the valuation of its regular part.

Examples

1) $p(x) = x + 3x^3 + 5x^5 \Rightarrow Val(p) = 1$.

2) $Val(\sin x) = 1$

3) $Val(\sin x - x) = 3$

4) $Val(e^x - 1) = 1$.

Property: If $Val(f) = p$ and $Val(g) = q$, the regular part to order $n$ of $f.g$ is obtained by multiplying the regular parts of $f$ and $g$ to order $n-q$ and $n-p$ respectively, and retaining only the terms of degree less than or equal to $n$.

Example: F.C(7/0) of $(\cos x - 1)(\sin x - x)$

$Val(\cos x - 1) = 2$ so it's enough to find the f.C5(0) of $\sin x - x$

$Val(\sin x - x) = 3$ so it's enough to find the f.C4(0) of $\cos x - 1$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4 \varepsilon(x)$$

$$\Rightarrow \cos x - 1 = \frac{-x^2}{2} + \frac{x^4}{24} + x^4 \varepsilon(x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5 \varepsilon(x)$$

$$\sin x - x = \frac{-x^3}{6} + \frac{x^5}{120} + x^5 \varepsilon(x)$$

$$(\cos x - 1)(\sin x - x) = \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\left(-\frac{x^3}{6} + \frac{x^5}{120}\right) + x^7 \varepsilon(x)$$

$$= \frac{x^5}{12} - \frac{x^7}{240} - \frac{x^7}{144} + x^7 \varepsilon(x)$$

$$(\cos x - 1)(\sin x - x) = \frac{x^5}{12} - \frac{1}{90} x^7 + x^7 \varepsilon(x)$$

## Division

If $f$ and $g$ have finite expansions to order $n$ near $0$ with $g(0) \neq 0$

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \varepsilon(x)$ and $g(x) = b_0 + b_1 x + \dots + b_n x^n + x^n \varepsilon(x)$

then so as for $\frac{f}{g}$, and it's finite expansion can be obtained using the euclidean division of the principal parts of the finite expansions of $f(x)$ and $g(x)$ but starting from the lowest power of $x$ and by retaining the powers of $x$ which are less than or equal to $n$.

**Ex** $F.E_3(0)$ of $\tan x$.

$$\tan x = \frac{\sin x}{\cos x}, \quad \cos 0 = 1 \neq 0.$$

$$\begin{array}{r|l} x - \frac{x^3}{6} & 1 - \frac{x^2}{2} \\ \hline - \quad x - \frac{x^3}{2} & x + \frac{x^3}{3} \\ \hline \frac{x^3}{3} & \end{array}$$

$$\sin x = x - \frac{x^3}{6} + x^3 \varepsilon(x)$$

$$\cos x = 1 - \frac{x^2}{2} + x^3 \varepsilon(x).$$

$$\text{so } \tan x = x + \frac{x^3}{3} + x^3 \varepsilon(x).$$

**Ex2** $F.E_2(0)$ of $\frac{\sin x}{x}$

$$\frac{\sin x}{x} = \frac{f(x)}{g(x)}$$ where $f(x) = \sin x$ and $g(x) = x$

$g(0) = 0$ and $Val(g) = 1$, so we make $F.E_3(0)$ of $\sin x$.

$$\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{6} + x^3\varepsilon(x)}{x} = 1 - \frac{x^2}{6} + x^2\varepsilon(x)$$

Remark: to find $F.e_n(0)$ of $\frac{f(x)}{g(x)}$ with $g(0)=0$. If $Val(g)=p$

we find the $f.e_{n+p}(0)$ of $f(x)$ and $g(x)$.

Composition: If $f$ and $g$ have finite expansions to order $n$

near $0$ with $g(0)=0$ and:

$$f(x) = P_n(x) + x^n\varepsilon(x), \quad g(x) = q_n(x) + x^n\varepsilon(x)$$

then $f\circ g$ has a finite expansion to order $n$ near $0$ and

it's principal part can be obtained from the expression

$P_n(q_n(x))$ by only retaining the powers of $x$ which are

less than or equal to $n$.

Remark: It's very important to notice that $g(0)=0$ in

order to find the finite expansion near $0$ of $f\circ g$ by

using the finite expansion of $f$ near $0$.

Example 1: F.e$_3$(0) of $\ln(1+\sin x)$

$$\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$\ln(1+\sin x) = \ln\left(1 + x - \frac{x^3}{6} + x^3\varepsilon(x)\right) = \ln(1+t) \text{ where}$$

$$t = x - \frac{x^3}{6} + x^3\varepsilon(x) \xrightarrow[x\to 0]{} 0$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + t^3\varepsilon(t)$$

$$\ln(1+\sin x) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 + x^3\varepsilon(x)$$

$$= x - \frac{x^3}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3\varepsilon(x)$$

$$\ln(1+\sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\varepsilon(x)$$

Example 2: F.e$_3$(0) of $e^{\tan x}$.

$$\tan x = x + \frac{1}{3}x^3 + x^3\varepsilon(x)$$

$$e^{\tan x} = e^{x + \frac{1}{3}x^3} = e^t \text{ where } t = x + \frac{x^3}{3} \xrightarrow[x\to 0]{} 0.$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + t^3\varepsilon(t)$$

$$e^{\tan x} = 1 + \left(x + \frac{x^3}{3}\right) + \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \frac{1}{6}\left(x + \frac{x^3}{3}\right)^3 + x^3\varepsilon(x)$$

$$= 1 + x + \frac{x^3}{3} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\varepsilon(x)$$

$$e^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + x^3\varepsilon(x)$$

Example 3 F.C4(0) of $Ln(1+\cos x)$.

$$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{4!}+x^4\varepsilon(x) = 1-\frac{x^2}{2}+\frac{x^4}{24}+x^4\varepsilon(x).$$

$$Ln(1+\cos x) = Ln\left(1+1-\frac{x^2}{2}+\frac{x^4}{24}+x^4\varepsilon(x)\right) = Ln\left(2-\frac{x^2}{2}+\frac{x^4}{24}+x^4\varepsilon(x)\right)$$

$$= Ln\left(2\left(1-\frac{x^2}{4}+\frac{x^4}{48}+x^4\varepsilon(x)\right)\right)$$

$$= Ln 2 + Ln\left(1-\frac{x^2}{4}+\frac{x^4}{48}+x^4\varepsilon(x)\right)$$

$$= Ln 2 + Ln(1+t) \text{ where } t = \frac{-x^2}{4}+\frac{x^4}{48} \xrightarrow[x\to 0]{} 0$$

$$Ln(1+t) = t-\frac{t^2}{2}+\cancel{\frac{t^3}{3}}-\cancel{\frac{t^4}{4}}+t^4\varepsilon(t)$$

$$Ln(1+\cos x) = Ln 2 + \left(\frac{-x^2}{4}+\frac{x^4}{48}\right)-\frac{1}{2}\left(\frac{-x^2}{4}+\frac{x^4}{48}\right)^2+x^4\varepsilon(x)$$

$$= Ln 2 - \frac{x^2}{4}+\frac{x^4}{48}-\frac{1}{32}x^4+x^4\varepsilon(x)$$

$$Ln(1+\cos x) = Ln 2 - \frac{x^2}{4}-\frac{1}{96}x^4+x^4\varepsilon(x).$$

Remark. $Ln(a+t) = Ln(a(1+t/a)) = Ln a + Ln(1+t/a)$ $a>0$

Example 4. F C2(0) of $e^{\cos x}$

$$\cos x = 1-\frac{x^2}{2}+x^2\varepsilon(x)$$

$$e^{\cos x} = e^{1-\frac{x^2}{2}+x^2\varepsilon(x)} = e^1 \cdot e^{-\frac{x^2}{2}+x^2\varepsilon(x)}$$

$e^{-\frac{x^2}{2}} = e^t$ where $t = -\frac{x^2}{2} \xrightarrow[x \to 0]{} 0$

$e^t = 1+t+t\varepsilon(t) \Rightarrow e^{-\frac{x^2}{2}} = 1-\frac{x^2}{2}+x^2\varepsilon(x)$

$\Rightarrow e^{\cos x} = e^1 \cdot \left(1-\frac{x^2}{2}\right)+x^2\varepsilon(x)$

$e^{\cos x} = e - \frac{e}{2}x^2 + x^2\varepsilon(x)$.

Remark. 1) $e^{a+t} = e^a \cdot e^t$

2) $\sqrt{a+t} = \sqrt{a(1+t/a)} = \sqrt{a} \cdot \sqrt{1+t/a}$ $a > 0$.

Remark $\frac{1}{1-x} = 1+x+x^2+x^3+\dots+x^n+x^n\varepsilon(x)$

so $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1-x+x^2-x^3+\dots+(-1)^n x^n + x^n\varepsilon(x)$.

Integration. If $f$ has a finite expansion to order $n$ near $0$, $f(x) = a_0+a_1x+\dots+a_nx^n+x^n\varepsilon(x)$ and if $F$ is a primitive of $f$ defined over an open interval $I$ containing $0$, then $F$ has a finite expansion to order $n+1$ near $0$ and it's given by:

$$F(x) = F(0) + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots + \frac{a_n}{n+1}x^{n+1} + x^{n+1}\varepsilon(x)$$

Example: Find the finite expansion to order 5 near 0 of arctan

Sol $f(x) = \arctan x \Rightarrow f'(x) = \dfrac{1}{1+x^2}$

we find the finite expansion to order 4 near 0 of $f'(x)$

$$\frac{1}{1+x^2} = \frac{1}{1+t} \quad \text{where } t = x^2 \xrightarrow[x \to 0]{} 0$$

$$= 1 - t + t^2 + t^2\varepsilon(t)$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + x^4\varepsilon(x)$$

then by integrating the principal part we get:

$$\arctan x = \arctan 0 + x - \frac{x^3}{3} + \frac{x^5}{5} + x^5\varepsilon(x)$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + x^5\varepsilon(x).$$

## Tangent to a curve:

Suppose $f$ has a finite expansion to order $n \geq 1$ near $0$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \varepsilon(x)$$

then the equation of the tangent $(T)$ to the representative curve $(C)$ of $f$ at $x=0$ is given by:

$$Y_T = a_0 + a_1 x$$

Moreover the relative position of the tangent $(T)$ and the curve $(C)$ near $x=0$ can be deduced from the finite expansion of $f$ by looking at the sign of the expression $f(x) - Y_T$ near $0$

$$f(x) - Y_T = a_2 x^2 + a_3 x^3 + \dots + a_n x^n + x^n \varepsilon(x)$$

for $x$ very close to $0$ ~~is~~.

$$f(x) - Y_T = a_2 x^2 + a_3 x^3 + \dots + a_n x^n + x^n \varepsilon(x)$$

The sign of the expression for $x$ very close to $0$, is given by the sign of it's dominant term, i.e the sign of the monomial of lowest power of $x$

For instance if the coefficient $a_2$ is different from $0$, then the dominant term is $a_2 x^2$ and we have, for $x$ near $0$

$$f(x) - Y_T \simeq a_2 x^2.$$

in this case, the sign of $f(x) - Y_T$ is the same as the sign of the constant $a_2$. therefore

- If $a_2 > 0$, then $f(x) - Y_T \geq 0$ near $0$ and the curve $(C)$ is above the tangent $(T)$.

- and if $a_2 < 0$, then $f(x) - Y_T \leq 0$ near $0$ and the curve $(C)$ is below the tangent $(T)$.

However if $a_2 = 0$, then we look at the next term in the finite expansion of $f(x)$ and we study it's sign.

Example 1: Suppose $f(x) = 1 - x + 6x^2 + x^2 \varepsilon(x)$. then the equation of the tangent $(T)$ at $x = 0$ is given by $Y_T = 1 - x$

Moreover, since $f(x) - Y_T \simeq 6x^2 \geq 0$ for all $x$ near $0$, the curve $(C)$ is above $(T)$ in the neighborhood of $(0, 1)$.

Example 2: Suppose now $f(x) = 2 + x - x^3 + x^3 \varepsilon(x)$. then the equation of the tangent $(T)$ at $x = 0$ is $Y_T = 2 + x$. Moreover,

$$f(x) - Y_T \simeq -x^3 \text{ then}$$

· For $x > 0$ near 0, we have

$f(x) - y_T < 0$ then the curve (C) is below (T) near 0.

· For $x < 0$ near 0, we have

$f(x) - y_T > 0$, then the curve (C) is above (T) near 0.

Exercise: Let $f$ be the function defined by:

$$f(x) = \frac{\ln(\operatorname{ch} x) + x + 1 - e^{\tan x}}{x^3}.$$

1) Give the $f.e_1(0)$ of $f(x)$.

2) Show that $f$ is extendable by continuity at $x = 0$ and give it's extension $g$.

3) show that $g$ is differentiable at 0 and give $g'(0)$.

4) give the equation of the tangent (T) to the curve (C) of $g$ at the point of abscissa 0.

Sol · $f.e_1(0)$ of $f$.

$\text{Val}(x^3) = 3$, so find $f.e_4(0)$ of $\ln(\operatorname{ch} x) + x + 1 - e^{\tan x}$.

f. $c_4(0)$ of $\ln(\operatorname{ch} x)$

$$\operatorname{ch} x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + x^4\varepsilon(x) \Rightarrow \operatorname{ch} x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + x^4\varepsilon(x)$$

$$\ln(\operatorname{ch} x) = \ln\left(1 + \frac{x^2}{2} + \frac{x^4}{24}\right) = \ln(1+t) \text{ where } t = \frac{x^2}{2} + \frac{x^4}{24} \xrightarrow[x \to 0]{} 0$$

$$\ln(1+t) = t - \frac{t^2}{2} + t^2\varepsilon(t)$$

$$\ln(\operatorname{ch} x) = \left(\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + x^4\varepsilon(x)$$

$$= \frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(\frac{x^4}{4}\right) + x^4\varepsilon(x)$$

$$\Rightarrow \boxed{\ln(\operatorname{ch} x) = \frac{x^2}{2} - \frac{x^4}{12} + x^4\varepsilon(x)}$$

• F. $c_4(0)$ of $e^{\tan x}$

$$\tan x = x + \frac{x^3}{3} + x^4\varepsilon(x).$$

$$e^{\tan x} = e^{x + \frac{x^3}{3}} = e^t \text{ when } t = x + \frac{x^3}{3} \xrightarrow[x \to 0]{} 0$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + t^4\varepsilon(t)$$

$$e^{\tan x} = 1 + \left(x + \frac{x^3}{3}\right) + \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \frac{1}{6}\left(x + \frac{x^3}{3}\right)^3 + \frac{1}{24}\left(x + \frac{x^3}{3}\right)^4 + x^4\varepsilon$$

$$= 1 + x + \frac{x^3}{3} + \frac{1}{2}\left(x^2 + \frac{2x^4}{3}\right) + \frac{1}{6}x^3 + \frac{1}{24}x^4 + x^4\varepsilon(x)$$

$$e^{\tan x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + x^4\varepsilon(x)$$

$$\text{So } f(x) = \frac{\frac{1}{2}x^2 - \frac{1}{12}x^4 + x + 1 - 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{3}{8}x^4 + x^4\varepsilon(x)}{x^3}$$

$$f(x) = \frac{\frac{-1}{2}x^3 - \frac{11}{24}x^4 + x^4\varepsilon(x)}{x^3}$$

$$\Rightarrow \boxed{f(x) = \frac{-1}{2} - \frac{11}{24}x + x\varepsilon(x)}$$

2) $\lim_{x \to 0} f(x) = a_0 = \frac{-1}{2}$ so $f$ is extendable by continuity at $x = 0$ by
$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \\ \frac{-1}{2} & \text{if } x = 0. \end{cases}$$

3) As $f$ has $f.e_1(0)$ then $g$ is differentiable at $0$ and
$$g'(0) = a_1 = \frac{-11}{24}$$

4) $(T)$ $y = \frac{-1}{2} - \frac{11}{24}x.$

Exercise: Let $f$ be the function defined by:
$$f(x) = \frac{x\ln(1+2x) + \operatorname{Sh}(ax)}{x}$$

1) give $f.e_3(0)$ of $f(x)$.

2) Show that $f$ is extendable by continuity at $x=0$. Give it's extension $g$.

3°) Determine the equation of the tangent (T) to the curve (C) of this extended function at the point A of abscissa $x=0$ of (C).

4°) Let $a = \sqrt[3]{12}$. Study the relative position of (C) and (T) near A.

Sol: 1. $\ell_3(0)$ of $f(x)$

$V_0(x) = 1 \Rightarrow$ 1. $\ell_4(0)$ of $x \ln(1+2x) + sh(ax)$

1. $\ell_4(0)$ of $x\ln(1+2x)$. as $V_0(x)=1 \Rightarrow$ f. $\ell_3(0)$ of $\ln(1+2x)$

$\ln(1+2x) = \ln(1+t)$ where $t = 2x \xrightarrow[x \to 0]{} 0$.

$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + t^3 \varepsilon(t)$

$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} + x^3 \varepsilon(x)$

$\Rightarrow \ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + x^3\varepsilon(x)$

$\Rightarrow x\ln(1+2x) = 2x^2 - 2x^3 + \frac{8}{3}x^4 + x^4\varepsilon(x)$

$sh(ax) = sh(t)$ where $t = ax \xrightarrow[x \to 0]{} 0$

$sh\,t = t + \frac{t^3}{6} + t^4\varepsilon(t)$

$sh(ax) = ax + \frac{a^3}{6}x^3 + x^4\varepsilon(x)$

$$\Rightarrow f(x) = \frac{2x^2 - 2x^3 + \frac{8}{3}x^4 + ax + \frac{a^3}{6}x^3 + x^4\varepsilon(x)}{x}$$

$$\Rightarrow f(x) = \frac{ax + 2x^2 + \left(\frac{a^3}{6} - 2\right)x^3 + \frac{8}{3}x^4 + x^4\varepsilon(x)}{x}$$

$$\Rightarrow \boxed{f(x) = a + 2x + \left(\frac{a^3}{6} - 2\right)x^2 + \frac{8}{3}x^3 + x^3\varepsilon(x)}$$

2) $\lim\limits_{x \to 0} f(x) = a$ so $f$ is extendable by continuity at $x = 0$ by $g(x) = \begin{cases} f(x) & \text{if } x \in D_f \\ a & \text{if } x = 0 \end{cases}$

3) $(T) : y = a + 2x$.

4) $a = \sqrt[3]{12} \Rightarrow \frac{a^3}{6} - 2 = 0$

$$\Rightarrow f(x) = \sqrt[3]{12} + 2x + \frac{8}{3}x^3 + x^3\varepsilon(x)$$

$y_T = \sqrt[3]{12} + 2x$ $(T)$.

$f(x) - y_T \simeq \frac{8}{3}x^3$

$x \to 0^+$: $> 0$ (C) is above (T) near 0

$x \to 0^-$: $< 0$ (C) is below (T) near 0.

Exercise: Let $f$ be the function defined by:

$$f(x) = \frac{\sin^2 x \cdot \ln(1+x)}{(1-e^{2x})\cdot(\sqrt{1+x^2}-1)}$$

1) Give the $f.e_2(0)$ of $f(x)$.

2) Show that $f$ is extendable by continuity at $x=0$. Let $g$ be it's extension.

3) Show that $g$ is differentiable at $0$ and give $g'(0)$.

4) Determine the equation of the tangent $(T)$ to the curve $(C)$ of $g$ at $x=0$ and precise the relative position of $(T)$ with respect to $(C)$ near $0$.

Sol: $Val(1-e^{2x}) = 1$, $Val(\sqrt{1+x^2}-1) = 2$ so

$$Val\left((1-e^{2x})(\sqrt{1+x^2}-1)\right) = 3.$$

$\Rightarrow f.e_5(0)$ of $\sin^2 x \cdot \ln(1+x)$ and $f.e_5(0)$ of $(1-e^{2x})(\sqrt{1+x^2}-1)$

$Val(\sin^2 x) = 2 \Rightarrow f.e_3(0)$ of $\ln(1+x)$

$Val(\ln(1+x)) = 1 \Rightarrow f.e_4(0)$ of $\sin^2 x$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\varepsilon(x).$$

$$\sin x = x - \frac{x^3}{6} + x^4\varepsilon(x)$$

$$\sin^2 x = \left(x - \frac{x^3}{6}\right)^2 + x^4\varepsilon(x) = x^2 - \frac{x^4}{3} + x^4\varepsilon(x).$$

$$\sin^2 x \cdot \ln(1+x) = \left(x^2 - \frac{x^4}{3}\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + x^5\varepsilon(x).$$

$$= x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \frac{x^5}{3} + x^5\varepsilon(x)$$

$$\Rightarrow \boxed{\sin^2 x \cdot \ln(1+x) = x^3 - \frac{x^4}{2} + x^5\varepsilon(x)}$$

. $f.e_5(0)$ of $f(1-e^{2x}).(\sqrt{1+x^2}-1)$

$Val((1-e^{2x}) = 1 \Rightarrow f.e_4(0)$ of $\sqrt{1+x^2}-1$

$Val(\sqrt{1+x^2}-1) = 2 \Rightarrow f.e_3(0)$ of $1-e^{2x}$.

$e^{2x} = e^t$ where $t = 2x \to 0$, $x \to 0$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + t^3\varepsilon(t)$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + x^3\varepsilon(x)$$

$$\Rightarrow \boxed{1 - e^{2x} = -2x - 2x^2 - \frac{4}{3}x^3 + x^3\varepsilon(x)}$$

$\sqrt{1+x^2} = \sqrt{1+t}$ where $t = x^2 \to 0$, $x \to 0$.

$$\sqrt{1+t} = (1+t)^{\frac{1}{2}} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + t^2\varepsilon(t)$$

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{1}{8}x^4 + x^4\varepsilon(x)$$

$$\Rightarrow \sqrt{1+x^2} - 1 = \frac{x^2}{2} - \frac{1}{8}x^4 + x^4\varepsilon(x)$$

$$(1-e^{2x})(\sqrt{1+x^2}-1) = \left(-2x - 2x^2 - \frac{4}{3}x^3\right)\left(\frac{x^2}{2} - \frac{x^4}{8}\right) + x^5\varepsilon(x)$$

$$= -x^3 + \frac{x^5}{4} - x^4 - \frac{2}{3}x^5 + x^5\varepsilon(x)$$

$$\Rightarrow (1-e^{2x})(\sqrt{1+x^2}-1) = -x^3 - x^4 - \frac{5}{12}x^5 + x^5\varepsilon(x)$$

$$\Rightarrow f(x) = \frac{x^3 - \frac{x^4}{2} + x^5\varepsilon(x)}{-x^3 - x^4 - \frac{5}{12}x^5 + x^5\varepsilon(x)} = \frac{1 - \frac{1}{2}x + x^2\varepsilon(x)}{-1 - x - \frac{5}{12}x^2 + x^5\varepsilon(x)}$$

$$\begin{array}{r|l}
\cancel{1} - \frac{1}{2}x & -1 - x - \frac{5}{12}x^2 \\ \cline{2-2}
- \quad +\cancel{1} + x + \frac{5}{12}x^2 & -1 + \frac{3}{2}x - \frac{13}{12}x^2 \\ \cline{1-1}
-\frac{3}{2}x - \frac{5}{12}x^2 & \\
- \quad -\frac{3}{2}x - \frac{3}{2}x^2 & \\ \cline{1-1}
+\frac{13}{12}x^2 &
\end{array}$$

$\Rightarrow f(x) = -1 + \frac{3}{2}x - \frac{13}{12}x^2 + x^2\varepsilon(x)$.

2) Since $f$ have $f.e_1(0)$ so $f$ is extendable by continuity at $x=0$ and $g(x) = \begin{cases} f(x) & \text{if } x \in D_f \\ -1 & \text{if } x = 0 \end{cases}$

3) Since $f$ has $f.e_1(0)$ then $g$ is differentiable at $0$ and $g'(0) = a_1 = 3/2$.

4) $(T): y = -1 + \frac{3}{2}x$

$f(x) - y \simeq \frac{-13}{12}x^2 < 0$ $(C_g)$ is below $(T)$ near $0$.