primitive:

(Antiderivative)

Definitions and notations:

Def: Let f be a function defined on an interval I of $\mathbb{R}$. we say that G is a primitive of f on I if G is differentiable on I and $G'(x) = f(x)$ $\forall x \in \mathbb{R}$.

Examples:

1) $f(x) = x^2 \Rightarrow G(x) = \frac{x^3}{3} + C, \quad C \in \mathbb{R}.$

2) $f(x) = \sin x \Rightarrow G(x) = -\cos x + C, \quad C \in \mathbb{R}.$

theorem: Let $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function and Let G be a primitive of f on I. the set of all primitives of f on I is of the form: $G(x) + C$, where C is an arbitrary constant in $\mathbb{R}$.

$F'(x) = f(x)$ $\quad \forall x \in I \Rightarrow F' = G' \Rightarrow F = G + C$.

$G'(x) = f(x)$

we denote by $\int f(x)dx = G(x) + C$

propaties:

1) $\int \alpha f(x)dx = \alpha \int f(x)dx$, $\forall \alpha \in \mathbb{R}$.

2) $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$.

theorem: Every continuous function on an interval I has primitives on I.

primitives of the usual functions

1) $\int a\,dx = ax + c$, $c \in \mathbb{R}$.

2) $\int x^{\alpha} dx = \dfrac{x^{\alpha+1}}{\alpha+1} + c$, $\alpha \neq -1$.

3) $\int \dfrac{dx}{x} = \ln|x| + c$.

4) $\int e^{ax} dx = \dfrac{1}{a} e^{ax} + c$, $a \neq 0$

5) $\int \sin(ax)\,dx = \dfrac{-1}{a}\cos(ax) + c$, $a \neq 0$.

6) $\int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$, $a \neq 0$

7) $\int \dfrac{dx}{\cos^2 x} = \int (1+\tan^2 x)\,dx = \tan x + c$.

8) $\int \dfrac{dx}{\sin^2 x} = -\cotan x + c$.

9) $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\dfrac{x}{a}\right) + c$, $a \neq 0$

10) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C, \ a \neq 0.$

11) $\int ch(ax)\,dx = \frac{1}{a} sh(ax) + C, \ a \neq 0$

12) $\int sh(ax)\,dx = \frac{1}{a} ch(ax) + C, \ a \neq 0$

13) $\int \frac{dx}{ch^2 x} = \int (1 - th^2 x)\,dx = th\, x + C.$

14) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$

15) $\int \frac{dx}{\sqrt{x^2+K}} = \ln\left|x + \sqrt{x^2+K}\right| + C.$

change of Variables:

theorem: Let $f: [a,b] \longrightarrow \mathbb{R}$ be a continuous function and let $\varphi: [c,d] \longrightarrow \mathbb{R}$ be a function of class C^1 on $[c,d]$ such that $\varphi([c,d]) \subseteq [a,b]$ then:

$$\int f(\varphi(x)).\, \varphi'(x)\,dx = \int f(t)\,dt$$

where $t = \varphi(x)$ and $dt = \varphi'(x)\,dx$.

Examples:

1) $I = \int \frac{x}{x^2+1}\,dx.$

$t = x^2 + 1 \Rightarrow dt = 2x\,dx$

$\Rightarrow I = \frac{1}{2} \int \frac{2x\,dx}{1+x^2} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + c$.

$\Rightarrow I = \frac{1}{2} \ln(x^2+1) + c$.

2) $I = \int x e^{x^2} dx$.

Let $t = x^2 \Rightarrow dt = 2x\,dx$

$I = \frac{1}{2} \int e^{x^2} \cdot (2x\,dx) = \frac{1}{2} \int e^t dt = \frac{1}{2} e^t + c$.

$\Rightarrow I = \frac{1}{2} e^{x^2} + c$.

3) $I = \int \tan x\,dx = \int \frac{\sin x}{\cos x} dx$.

$t = \cos x \Rightarrow dt = -\sin x\,dx$

$\Rightarrow I = -\int \frac{(-\sin x)dx}{\cos x} = -\int \frac{dt}{t} = -\ln|t| + c$.

$\Rightarrow I = -\ln|\cos x| + c$.

4) $I = \int \frac{\ln x}{x} dx \quad , \quad x > 0$

$t = \ln x \Rightarrow dt = \frac{dx}{x}$

$\Rightarrow I = \int t\,dt = \frac{t^2}{2} + c = \frac{\ln^2 x}{2} + c$.

5) $I = \int \cos^3 x . \sin x \, dx$

$t = \cos x \Rightarrow dt = -\sin x \, dx$

$\Rightarrow I = -\int \cos^3 x . (-\sin x \, dx) = -\int t^3 dt = -\frac{t^4}{4} + c.$

$\Rightarrow I = -\frac{\cos^4 x}{4} + c.$

or: $t = \sin x \Rightarrow dt = \cos x dx.$

$\Rightarrow I = \int \cos^2 x . \sin x . \cos x dx = \int (1 - \sin^2 x) . \sin x . \cos x dx$

$\Rightarrow I = \int (1 - t^2) . t dt = \int (t - t^3) dt = \frac{t^2}{2} - \frac{t^4}{4} + c.$

$\Rightarrow I = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + c$

6) $I = \int (\tan x + \tan^3 x) dx$

$I = \int \tan x (1 + \tan^2 x) dx.$ Let $t = \tan x \Rightarrow dt = (1 + \tan^2 x) dx$

$\Rightarrow I = \int t dt = \frac{t^2}{2} + c = \frac{\tan^2 x}{2} + c.$

7) $I = \int \frac{1}{\cos^2 x} . e^{\tan x} dx.$

$t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x} dx$

$\Rightarrow I = \int e^t dt = e^t + c = e^{\tan x} + c.$

Integration by parts:

Theorem: Let u and v be two functions of class C^1 on I

then $\int u v' dx = uv - \int u' v \, dx$

Examples:

1) $I = \int x e^x dx$.

$u = x \Rightarrow u' = 1$.

$v' = e^x \Rightarrow v = e^x$

$\Rightarrow I = x e^x - \int 1 . e^x dx = x e^x - e^x + c = (x-1) e^x + C$.

2) $I = \int x \ln x \, dx$

$u = \ln x \Rightarrow u' = \frac{1}{x}$.

$v' = x \Rightarrow v = \frac{x^2}{2}$

$\Rightarrow I = \frac{x^2}{2} \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$\Rightarrow I = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$.

3) $I = \int \arctan(x) dx$

$u = \arctan(x) \Rightarrow u' = \frac{1}{1+x^2}$

$v' = 1 \Rightarrow v = x$.

$\Rightarrow I = x \arctan x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2} = x \arctan x - \frac{1}{2}\ln(1+x^2) + C.$

Type A: Let $p(x)$ be a polynomial of degree n, $n \in \mathbb{N}$.

to calculate:

1) $I_1 = \int p(x) \cdot e^{ax}\, dx$

2) $I_2 = \int p(x) \cdot \sin(ax)\,dx$

3) $I_3 = \int p(x) \cdot \cos(ax)\,dx$

4) $I_4 = \int p(x) \cdot sh(ax)\,dx$

5) $I_5 = \int p(x) \cdot ch(ax)\,dx$

6) $I_6 = \int p(x) \cdot (1+ax)^{\alpha}\,dx$, $\alpha \in \mathbb{R}$.

we integrate n times by parts and we get $u(x) = p(x)$.

Example: $I = \int x^2\, ch x\, dx$.

$u(x) = x^2 \Rightarrow u' = 2x$

$v' = ch\, x \Rightarrow v = sh\, x$

$$\Rightarrow I = x^2 sh\, x - \underbrace{\int 2x\, sh\, x\, dx}_{J}$$

$J = \int 2x\, sh\, x\, dx.$

$u = 2x \Rightarrow u' = 2$

$v' = sh\, x \Rightarrow v = ch\, x$

$\Rightarrow J = 2x\,chx - \int 2chx\,dx = 2x\,chx - 2shx + c.$

$\Rightarrow I = x^2 shx - 2x\,chx + 2shx + c.$

Remark: to calculate an integral of type A, we can use the following algorithm:

derivative		integral
x^2		chx
	$+$	
$2x$		shx
	$-$	
2		chx
	$+$	
0		shx

$\Rightarrow I = x^2 shx - 2x\,chx + 2shx + c.$

Example 2 $\quad J = \int (x^3 + x + 1)\sin(3x)\,dx$

$x^3 + x + 1$		$\sin(3x)$
	$+$	
$3x^2 + 1$		$\frac{-1}{3}\cos(3x)$
	$-$	
$6x$		$\frac{-1}{9}\sin(3x)$
	$+$	
6		$\frac{1}{27}\cos(3x)$
	$-$	
0		$\frac{1}{81}\sin(3x)$

$$\Rightarrow I = \frac{-1}{3}(x^3 + x + 1)\cos(3x) + \frac{1}{9}(3x^2+1)\sin(3x)$$

$$+ \frac{6x}{27}\cos(3x) - \frac{6}{81}\sin(3x) + C.$$

Type B: Let $p(x)$ be a polynomial of degree n, $n \in \mathbb{N}$

to calculate:

1) $I_1 = \int p(x) \cdot \ln(ax)\,dx$

2) $I_2 = \int p(x) \cdot \arcsin(ax)\,dx$

3) $I_3 = \int p(x) \cdot \arccos(ax)\,dx$

4) $I_4 = \int p(x)\, \operatorname{argsh}(ax)\,dx$

5) $I_5 = \int p(x)\, \arctan(ax)\,dx$

6) $I_6 = \int p(x) \cdot \operatorname{argch}(ax)\,dx$

7) $I_7 = \int p(x) \cdot \operatorname{argth}(ax)\,dx$

we integrate by parts and we get $V' = p(x)$.

Examples:

1) $I = \int x^2 \ln x\,dx$.

$u = \ln x \Rightarrow u' = \frac{1}{x}$

$v' = x^2 \Rightarrow v = \frac{x^3}{3}$

$$\Rightarrow I = \frac{x^3}{3}\ln x - \int \frac{1}{x}\cdot\frac{x^3}{3}dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

2) $I = \int x \arctan x \, dx$.

$u = \arctan x \Rightarrow u' = \frac{1}{1+x^2}$

$v' = x \Rightarrow v = \frac{x^2}{2}$.

$$\Rightarrow I = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$$\int \frac{x^2}{1+x^2}dx = \int \frac{(x^2+1)-1}{x^2+1}dx = \int \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= x - \arctan x + C.$$

$$\Rightarrow I = \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C.$$

type c to calculate:

1) $\int e^{ax}\cdot \sin(bx)dx$

2) $\int e^{ax}\cos(bx)dx$

3) $\int e^{ax}\cdot \mathrm{sh}(bx)dx$

4) $\int e^{ax}\cdot \mathrm{ch}(bx)dx$

we integrate by parts.

Example: Calculate $I = \int e^{-x} \sin x \, dx$.

$u(x) = e^{-x} \Rightarrow u' = -e^{-x}$

$V' = \sin x \Rightarrow V = -\cos x$.

$\Rightarrow I = -e^{-x} \cos x - \underbrace{\int e^{-x} \cos x \, dx}_{J}$

$\Rightarrow I = -e^{-x} \cos x - J$.

$J = \int e^{-x} . \cos x \, dx$

$u(x) = e^{-x} \Rightarrow u' = -e^{-x}$

$V' = \cos x \Rightarrow V = \sin x$

$\Rightarrow J = e^{-x} \sin x + \int e^{-x} \sin x \, dx = e^{-x} \sin x + I$

$\Rightarrow J = e^{-x} \sin x + I$

$\Rightarrow I = -e^{-x} \cos x - e^{-x} \sin x - I \Rightarrow 2I = -e^{-x}(\cos x + \sin x)$

$$\Rightarrow I = \frac{-e^{-x}(\sin x + \cos x)}{2} + C.$$

Exercise: Calculate the following integrals:

1) $I_1 = \int x . \frac{\cos x}{\sin x} dx$

2) $I_2 = \int \sin(\ln x) \, dx$

3) $I_3 = \int e^{\sin^2 x} . \sin(2x) \, dx$

4) $I_4 = \int x . \frac{\arcsin x}{\sqrt{1-x^2}} dx$.

Solution:

1) $I_1 = \int x \cdot \frac{\cos x}{\sin^3 x} dx$.

$u(x) = x \Rightarrow u' = 1$.

$v' = \frac{\cos x}{\sin^3 x} \Rightarrow v = \int \frac{\cos x}{\sin^3 x} dx = \int \frac{dt}{t^3} = \frac{t^{-3+1}}{-3+1} + C$

$= \frac{-1}{2t^2} + C$

Let $t = \sin x \Rightarrow dt = \cos x\, dx$

$\Rightarrow v = \frac{-1}{2\sin^2 x}$

$\Rightarrow I_1 = \frac{-x}{2\sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin^2 x} = \frac{-x}{2\sin^2 x} - \frac{1}{2}\cotan(x) + C$.

2) $I_2 = \int \sin(\ln x) dx$.

$u = \sin(\ln x) \Rightarrow u' = \frac{1}{x}\cos(\ln x)$

$v' = 1 \Rightarrow v = x$.

$\Rightarrow I_2 = x\sin(\ln x) - \int \cos(\ln x) dx$

$I_2 = x\sin(\ln x) - J, \quad J = \int \cos(\ln x) dx$

$u = \cos(\ln x) \Rightarrow u' = \frac{-1}{x}\sin(\ln x)$

$v' = 1 \Rightarrow v = x$.

$\Rightarrow J = x\cos(\ln x) + I_2$

$\Rightarrow I_2 = x\sin(\ln x) - x\cos(\ln x) - I_2$

$$\Rightarrow I_2 = \frac{x\left(\sin(\ln x) - \cos(\ln x)\right)}{2} + C.$$

3) $I_3 = \int e^{\sin^2 x} \cdot \sin(2x)\,dx.$

Let $t = \sin^2 x \Rightarrow dt = 2\sin x \cos x\,dx \Rightarrow dt = \sin(2x)\,dx.$

$\Rightarrow I_3 = \int e^t\,dt = e^t + C = e^{\sin^2 x} + C.$

4) $I_4 = \int x \cdot \frac{\arcsin x}{\sqrt{1-x^2}}\,dx.$

$u(x) = \arcsin x \Rightarrow u' = \frac{1}{\sqrt{1-x^2}}$

$v' = \frac{x}{\sqrt{1-x^2}} \Rightarrow v = \int \frac{x}{\sqrt{1-x^2}}\,dx = \frac{-1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = \frac{-1}{2}\int \frac{dt}{\sqrt{t}}$

$= \frac{-1}{2}\left[\frac{t^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right] = -\sqrt{t} = -\sqrt{1-x^2}$

where $t = 1-x^2.$

$\Rightarrow I_4 = -\sqrt{1-x^2}\arcsin x + \int 1\,dx$

$= -\sqrt{1-x^2}\arcsin x + x + C.$

$\Rightarrow I_4 = -\sqrt{1-x^2}\arcsin x + \int 1\,dx$

$\Rightarrow I_4 = -\sqrt{1-x^2}\arcsin x + x + C.$

Integrals of the form $I = \int \frac{P(x)}{Q(x)}dx$.

In this section, we study the integrals of the form

$$I = \int \frac{P(x)}{Q(x)}dx$$ where $P(x)$ and $Q(x)$ are

two real polynomials.

1^{st} case: $\deg(P) < \deg(Q)$.

a) $I_n = \int \frac{dx}{(x-a)^n}$ where $a \in \mathbb{R}$ and $n \in \mathbb{N}^*$.

For $n=1 \Rightarrow I_1 = \int \frac{dx}{x-a} = \ln|x-a| + C.$

For $n \geq 2 \Rightarrow I_n = \int (x-a)^{-n}dx = \frac{(x-a)^{-n+1}}{-n+1} + C$ $\quad (t = x-a \Rightarrow dt = dx)$.

Remark: $\int \frac{dx}{2x+1}$? Let $t = 2x+1 \Rightarrow dt = 2dx$

$\Rightarrow \int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{2dx}{2x+1} = \frac{1}{2}\ln|2x+1| + C.$

So $\int \frac{dx}{bx+c} = \frac{1}{b}\ln|bx+c| + C_1$, $b \neq 0$.

b) $I = \int \frac{\alpha x + \beta}{a x^2 + b x + c} dx$.

Example 1: $I = \int \frac{6x + 3}{x^2 + 2x + 1} dx$

$$I = 3\int \frac{2x+2}{x^2+2x+1} dx - 3 \underbrace{\int \frac{dx}{x^2+2x+1}}_{J} = 3 \ln|x^2+2x+1| - 3J$$

$$J = \int \frac{dx}{x^2+2x+1} = \int \frac{dx}{(x+1)^2} = \frac{-1}{x+1} + C.$$

$$\Rightarrow I = 3 \ln|x^2+2x+1| + \frac{3}{x+1} + C.$$

2) $I = \int \frac{2x+3}{x^2+x-2} dx$

$$I = \int \frac{2x+1}{x^2+x-2} dx + 2\int \frac{dx}{x^2+x-2} = \ln|x^2+x-2| + 2J$$

$$J = \int \frac{dx}{x^2+x-2}.$$

$$x^2 + x - 2 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - 2 = \left(x+\frac{1}{2}\right)^2 - \frac{9}{4} = \left(x+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$\Rightarrow J = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} = \frac{1}{2\left(\frac{3}{2}\right)} \ln\left|\frac{x+\frac{1}{2}-\frac{3}{2}}{x+\frac{1}{2}+\frac{3}{2}}\right| + C$$

$\Rightarrow J = \ln|x^2+x-2| + \frac{2}{3}\ln\left|\frac{x-1}{x+2}\right| + c.$

3) $I = \int \frac{2x+1}{x^2+4x+6}\,dx$

$$I = \int \frac{2x+4}{x^2+4x+6}\,dx - 3\underbrace{\int \frac{dx}{x^2+4x+6}}_{J} = \ln|x^2+4x+6| - 3J$$

$$J = \int \frac{dx}{x^2+4x+6} = \int \frac{dx}{(x+2)^2-4+6} = \int \frac{dx}{(x+2)^2+2} = \int \frac{dx}{(x+2)^2+(\sqrt{2})^2}$$

$$= \int \frac{dt}{t^2+a^2} \quad \text{where } t = x+2,\ a = \sqrt{2}$$

$$= \frac{1}{a}\arctan\left(\frac{t}{a}\right) + c$$

$$\Rightarrow J = \frac{1}{\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right) + c.$$

$$\Rightarrow I = \ln|x^2+4x+6| - \frac{3}{\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right) + c.$$

4) $I = \int \frac{3x}{x^2+x+1}\,dx.$

$$I = \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\underbrace{\int \frac{dx}{x^2+x+1}}_{J} = \frac{3}{2}\ln|x^2+x+1| - \frac{3}{2}J$$

$$J = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+\frac{1}{2})^2-\frac{1}{4}+1} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$$\Rightarrow I = \frac{3}{2} \operatorname{Ln}|x^2 + x + 1| - \frac{3}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{2}}\right) + C.$$

Decomposition into simple elements.

Def: We call simple elements, every element of the form $\frac{1}{(x-a)^n}$ or $\frac{\alpha x + \beta}{ax^2 + bx + c}$ s.t $\Delta = b^2 - 4ac < 0$.

How to decompose $\frac{P(x)}{Q(x)}$ into sum of simple elements.

1) $$\frac{x+1}{(x-1)^1 (x+2)^1 (x+3)^1} = \frac{a}{x-1} + \frac{b}{x+2} + \frac{c}{x+3}.$$

2) $$\frac{x+4}{(x-1)^3 (x+3)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{x+3}.$$

3) $$\frac{x+5}{(x-2)^2 (x^2+x+3)} = \frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{cx+d}{x^2+x+3}$$

4) $$\frac{x^2+1}{(x-3)^2 (x+2) (x^2+x+1)^2} = \frac{a}{x-3} + \frac{b}{(x-3)^2} + \frac{c}{(x+2)} + \frac{cx+\delta}{x^2+x+1} + \frac{\alpha x+\beta}{(x^2+x+1)^2}$$

Exercise: calculate the following integrals:

1) $$I = \int \frac{x+2}{(x-1)(x+4)} dx$$

$$\frac{x+2}{(x-1)(x+4)} = \frac{a}{x-1} + \frac{b}{x+4} \Rightarrow \frac{a(x+4)+b(x-1)}{(x-1)(x+4)} = \frac{x+2}{(x-1)(x+4)}$$

$$\Rightarrow a(x+4)+b(x-1) = x+2 \Rightarrow (a+b)x + 4a - b = x+2.$$

$$\Rightarrow \begin{cases} a+b=1 \\ 4a-b=2 \end{cases} \Rightarrow a = \frac{3}{5},\ b = \frac{2}{5}.$$

$$\Rightarrow I = \int \frac{\frac{3}{5}}{x-1}\,dx + \int \frac{\frac{2}{5}}{x+4}\,dx$$

$$= \frac{3}{5}\int \frac{dx}{x-1} + \frac{2}{5}\int \frac{dx}{x+4} = \frac{3}{5}\ln|x-1| + \frac{2}{5}\ln|x+4| + C$$

2) $I = \int \frac{dx}{x^3-1}$; $p(x) = 1$, $Q(x) = x^3-1$ with $\deg(p) < \deg(Q)$

$$x^3-1 = (x-1)(x^2+x+1)$$

$$\Rightarrow I = \int \frac{dx}{(x-1)(x^2+x+1)}.$$

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+1}$$

$$\Rightarrow a(x^2+x+1) + (bx+c)(x-1) = 1$$

$$\Rightarrow ax^2 + ax + a + bx^2 - bx + cx - c = 1$$

$$\Rightarrow (a+b)x^2 + (a-b+c)x + a - c = 1.$$

$$\Rightarrow \begin{cases} a+b=0 \\ a-b+c=0 \\ a-c=1. \end{cases} \Rightarrow a=\frac{1}{3},\ b=\frac{-1}{3},\ c=\frac{-2}{3}.$$

$$\Rightarrow I=\frac{1}{3}\int\frac{dx}{x-1}+\int\frac{\left(\frac{-1}{3}x-\frac{2}{3}\right)dx}{x^2+x+1}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x+2}{x^2+x+1}dx$$

$$\int\frac{x+2}{x^2+x+1}dx=\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx+\frac{3}{2}\underbrace{\int\frac{dx}{x^2+x+1}}_{J}$$

$$=\frac{1}{2}\ln|x^2+x+1|+\frac{3}{2}J$$

$$J=\int\frac{dx}{x^2+x+1}=\int\frac{dx}{(x+\frac{1}{2})^2-\frac{1}{4}+1}=\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}=\int\frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$=\frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$=\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$\Rightarrow I=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|x^2+x+1|-\frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

2nd case: $\deg(P)\geq\deg(Q)$.

To calculate $I=\int\frac{P(x)}{Q(x)}dx$ when $\deg(P)\geq\deg(Q)$

make an euclidean division of $P(x)$ by $Q(x)$

$$\begin{array}{l|l} P(x) & Q(x) \\ \cline{2-2} & A(x) \\ \cline{1-1} R(x) & \end{array}$$

$$\Rightarrow P(x) = A(x)\,Q(x) + R(x) \Rightarrow \frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

when $\deg(R) < \deg(Q)$.

$$\Rightarrow I = \int \frac{P(x)}{Q(x)}\,dx = \int A(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx \text{ when } \deg(R) < \deg(Q)$$

Ex calculate $I = \int \frac{x^3}{x^2+1}\,dx$.

$$\begin{array}{l|l} x^3 & x^2+1 \\ \cline{2-2} x^3+x & x \\ \cline{1-1} -x & \end{array}$$

$$\Rightarrow x^3 = x(x^2+1) - x$$

$$\Rightarrow \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$\Rightarrow I = \int \frac{x^3}{x^2+1}\,dx = \int \left(x - \frac{x}{x^2+1}\right) dx$$

$$= \int x\,dx - \int \frac{x\,dx}{x^2+1}$$

$$= \frac{x^2}{2} - \frac{1}{2}\int \frac{2x\,dx}{x^2+1}$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + c$$

Primitive of rational functions in sinx and cosx.

in this section, we study the integrals of the form:

$$I = \int \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)}\,dx.$$

particular case: $I = \int \sin^m(x) \cdot \cos^n(x)\,dx$ where $m, n \in \mathbb{N}$.

1) If m is odd $\Rightarrow m = 2k+1$

$\Rightarrow I = \int \sin^{2k+1}(x) \cdot \cos^n x\,dx = \int \sin^{2k}(x) \cdot \cos^n(x) \cdot \sin x\,dx$

Let $t = \cos x \Rightarrow dt = -\sin x\,dx$

$\Rightarrow I = -\int (\sin^2 x)^k \cdot \cos^n(x) \cdot (-\sin x)\,dx = -\int (1-\cos^2 x)^k \cdot \cos^n x \cdot (-\sin x\,dx)$

$\Rightarrow I = -\int (1-t^2)^k \cdot t^n\,dt.$

so if m is odd, we make the change of variables $t = \cos x$.

If n is odd, we make the change of variables $t = \sin x$.

. If m and n are even, we linearize using the following formula

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sin 2x = 2\sin x \cdot \cos x$$

Example 1: $I = \int \sin^3 x . \cos^2 x \, dx$

$t = \cos x \Rightarrow dt = -\sin x \, dx$

$\Rightarrow I = \int \sin^2 x . \cos^2 x . \sin x \, dx = -\int (1-\cos^2 x) \cdot \cos^2 x (-\sin x \, dx)$

$\Rightarrow I = -\int (1-t^2) . t^2 dt = -\int t^2 dt + \int t^4 dt$

$= -\frac{t^3}{3} + \frac{t^5}{5} + c$

$\Rightarrow I = \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + C .$

Example 2 $I = \int \cos^2 x \, dx .$

$I = \int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} dx = \int \frac{1}{2} dx + \frac{1}{2} \int \cos 2x \, dx$

$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C .$

Example 3: $I = \int \sin^2 x . \cos^2 x \, dx .$

$\sin x . \cos x = \frac{\sin(2x)}{2} \Rightarrow \sin^2 x . \cos^2 x = \frac{1}{4} \sin^2 (2x)$

$= \frac{1}{4} \left(\frac{1-\cos 4x}{2} \right) = \frac{1}{8} - \frac{1}{8} \cos 4x$

$\Rightarrow I = \frac{1}{8} \int dx - \frac{1}{8} \int \cos 4x \, dx = \frac{1}{8} x - \frac{1}{32} \sin 4x + C$

General case: (Bioche rules) $I = \int \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)} dx$

Let $A(x) = \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)} dx$.

1) If $A(-x) = A(x)$: we make the change of variables $t = \cos x$.

2) If $A(\pi - x) = A(x)$: " " " " " $t = \sin x$.

3) If $A(\pi + x) = A(x)$: " " " " " " " $t = \tan x$.

we can use the change of variables $t = \tan\left(\frac{x}{2}\right)$ using

the following formulas:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}$$

$$dt = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) dx = \frac{1}{2}(1+t^2) dx \Rightarrow dx = \frac{2dt}{1+t^2}.$$

In all cases, we obtain an integral of the form $\int \frac{h(t)}{L(t)} dt$

where h and L are two real polynomials.

Remark.

$\cos(-x) = \cos(x)$ | $\cos(\pi - x) = -\cos x$
$\sin(-x) = -\sin x$ | $\sin(\pi - x) = \sin x$
$\tan(-x) = -\tan x$ | $\tan(\pi - x) = -\tan x$
$d(-x) = -d(x)$ | $d(\pi - x) = -dx$

• $\cos(\pi+x) = -\cos x$; $\sin(\pi+x) = -\sin x$, $\tan(\pi+x) = \tan x$

and $d(\pi+x) = dx$.

Exercise: Calculate the following integrals:

1) $I = \int \frac{\sin x . \cos x}{1+\cos^2 x} dx$.

Let $A(x) = \frac{\sin x . \cos x}{1+\cos^2 x} dx$

$A(x) = \frac{-\sin x . \cos x . (-dx)}{1+\cos^2 x} = A(x) \Rightarrow t = \cos x \Rightarrow dt = -\sin x dx$

$\Rightarrow I = -\int \frac{\cos x . (-\sin x dx)}{1+\cos^2 x} = -\int \frac{t\,dt}{1+t^2} = \frac{-1}{2}\int \frac{2t\,dt}{1+t^2} = \frac{-1}{2} \ln(1+t^2) + c$

$\Rightarrow I = \frac{-1}{2} \ln(1+\cos^2 x) + c$.

2) $J = \int \frac{dx}{\sin x}$

$A(x) = \frac{dx}{\sin x} \Rightarrow A(-x) = A(x) \Rightarrow t = \cos x \Rightarrow dt = -\sin x dx$

$I = \int \frac{dx}{\sin x} = -\int \frac{-\sin x dx}{\sin^2 x} = -\int \frac{-\sin x dx}{1-\cos^2 x} = -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1}$

$= \frac{1}{2} \ln \left|\frac{t-1}{t+1}\right| + c = \frac{1}{2} \ln \left|\frac{\cos x - 1}{\cos x + 1}\right| + c$

3) $I = \int \frac{dx}{\sin^2 x \cdot \cos^2 x}$

$A(x) = \frac{dx}{\sin^2 x \cdot \cos^2 x} \Rightarrow A(\pi + x) = A(x)$

$t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x} dx$

$\Rightarrow I = \int \frac{1}{\sin^2 x} \frac{dx}{\cos^2 x} = \int \frac{1 + \tan^2 x}{\tan^2 x} \frac{dx}{\cos^2 x} = \int \frac{1+t^2}{t^2} dt \quad \left(\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}\right.$

$\Rightarrow I = \int \left(1 + \frac{1}{t^2}\right) dt = t - \frac{1}{t} + c = \tan x - \frac{1}{\tan x} + c$

4. $I = \int \frac{dx}{2 + \cos x}$

$t = \tan\left(\frac{x}{2}\right) \Rightarrow dx = \frac{2dt}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$

$\Rightarrow I = \int \frac{\frac{2dt}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} = 2 \int \frac{dt}{t^2+3} = 2 \int \frac{dt}{t^2 + (\sqrt{3})^2}$

$= \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}} \arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c$

5. $I = \int \frac{\cos^3 x}{\sin^4 x} dx$

$A(x) = \frac{\cos^3 x}{\sin^4 x} dx \Rightarrow A(\pi - x) = A(x) \Rightarrow t = \sin x \Rightarrow dt = \cos x\, dx$

$$I = \int \frac{\cos^2 x \cdot \cos x \, dx}{\sin^4 x} = \int \frac{(1-\sin^2 x)\cdot \cos x dx}{\sin^4 x} = \int \frac{1-t^2}{t^4} dt$$

$$= \int \frac{dt}{t^4} - \int \frac{dt}{t^2} = \frac{-1}{3t^3} + \frac{1}{t} + C = \frac{-1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

Integrals of rational functions in shx and chx.

In this section, we study the integral of the form:

$$I = \int \frac{P(shx, chx)}{Q(shx, chx)} dx$$

Bioche rule's:

$$A(x) = \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)} dx$$

If $A(-x) = A(x) \Rightarrow t = chx$

If $A(\pi - x) = A(x) \Rightarrow t = shx$

If $A(\pi + x) = A(x) \Rightarrow t = thx$

we can make the change of variables $t = th\frac{x}{2}$ using the following formulas:

$$t = th\frac{x}{2}, \quad shx = \frac{2t}{1-t^2}, \quad chx = \frac{1+t^2}{1-t^2}$$

$$dx = \frac{2dt}{1-t^2}$$

. we can write chx and shx in terms of e^x then, we make the change of variables $t = e^x$.

in all cases, we obtain an integral of the form

$\int \frac{h(t)}{L(t)}\,dt$ where h and L are two real polynomials.

Exercise: Calculate the following integrals.

1) $I = \int \frac{dx}{\operatorname{sh} x}$

1st Method: $A(x) = \frac{dx}{\sinh x} \Rightarrow A(-x) = A(x)$

Let $t = \operatorname{ch} x \Rightarrow dt = \operatorname{sh} x\, dx$

$\Rightarrow I = \int \frac{dx}{\operatorname{sh} x} = \int \frac{\operatorname{sh} x\, dx}{\operatorname{sh}^2 x} = \int \frac{\operatorname{sh} x\, dx}{\operatorname{ch}^2 x - 1} = \int \frac{dt}{t^2 - 1} = \ln\left|\frac{t-1}{t+1}\right| + c$

$\Rightarrow I = \ln\left|\frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}\right| + c.$

2nd Method: $I = \int \frac{dx}{\operatorname{sh} x} = \int \frac{dx}{\frac{e^x - e^{-x}}{2}} = 2\int \frac{dx}{e^x - \frac{1}{e^x}} = 2\int \frac{e^x dx}{e^{2x} - 1}$

Let $t = e^x \Rightarrow dt = e^x dx$

$\Rightarrow I = 2\int \frac{dt}{t^2 - 1} = \ln\left|\frac{t-1}{t+1}\right| + c = \ln\left|\frac{e^x - 1}{e^x + 1}\right| + c.$

2) $I = \int \frac{dx}{\operatorname{ch} x} = \int \frac{dx}{\frac{e^x + e^{-x}}{2}} = 2\int \frac{dx}{e^x + \frac{1}{e^x}} = 2\int \frac{e^x dx}{e^{2x} + 1}$

Let $t = e^x \Rightarrow dt = e^x dx$

$$\Rightarrow I = 2\int \frac{dt}{t^2+1} = 2\arctan(t) + c = 2\arctan(e^x) + c.$$

- Integrals of the form $I = \int \frac{f(e^x)}{g(e^x)} dx$

we make the change of variables $t = e^x \Rightarrow dt = e^x dx$
$\Rightarrow dt = t\,dx$

Ex $I = \int \frac{dx}{e^x+1}$

Let $t = e^x \Rightarrow dt = e^x dx$

$$I = \int \frac{dx}{e^x+1} = \int \frac{e^x dx}{e^x(e^x+1)} = \int \frac{dt}{t(t+1)}$$

$$\frac{1}{t(t+1)} = \frac{1+t-t}{t(1+t)} = \frac{1+t}{t(1+t)} - \frac{t}{t(1+t)} = \frac{1}{t} - \frac{1}{1+t}$$

$$\Rightarrow I = \int \frac{dt}{t} - \int \frac{dt}{1+t} = \ln|t| - \ln|1+t| + c$$

$$\Rightarrow I = \ln e^x - \ln(1+e^x) + c \Rightarrow I = x - \ln(1+e^x) + c$$

Integrals of the form $I = \int f\left(x, \sqrt{\frac{ax+b}{cx+d}}\right)dx$.

to calculate I, we make the change of variables $t = \sqrt{\frac{ax+b}{cx+d}}$

Example: calculate $I = \int \frac{1}{x} \cdot \sqrt{\frac{1-x}{1+x}}\, dx$

Let $t = \sqrt{\frac{1-x}{1+x}} \Rightarrow t^2 = \frac{1-x}{1+x} \Rightarrow 1-x = t^2 + t^2x \Rightarrow 1-t^2 = x + t^2x$

$\Rightarrow x(1+t^2) = 1-t^2 \Rightarrow x = \frac{1-t^2}{1+t^2}$

$\Rightarrow dx = \left(\frac{1-t^2}{1+t^2}\right)' dt = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} dt = \frac{-4t}{(1+t^2)^2} dt$

$\Rightarrow I = \int \frac{1}{x} \cdot \sqrt{\frac{1-x}{1+x}}\, dx = \int \frac{1+t^2}{1-t^2} \cdot t \cdot \frac{-4t}{(1+t^2)^2} dt$

$\Rightarrow I = -\int \frac{4t^2}{(1-t^2)(1+t^2)} dt$

$\frac{4t^2}{(1-t^2)(1+t^2)} = \frac{at+b}{1-t^2} + \frac{ct+d}{1+t^2}$

$(1-t^2)(1+t^2)$ ... $(at+b)(1+t^2) + (ct+d)(1-t^2) = 4t^2$

$\Rightarrow (at+b)(1+t^2) + (ct+d)(1-t^2) = 4t^2$

$\Rightarrow at + at^3 + b + bt^2 + ct - ct^3 + d - dt^2 = 4t^2$

$\Rightarrow (a-c)t^3 + (b-d)t^2 + (a+c)t + b + d = 4t^2$

$\Rightarrow \begin{cases} a-c=0 \\ b-d=4 \\ a+c=0 \\ b+d=0 \end{cases} \Rightarrow \begin{cases} a-c=0 \\ a+c=0 \\ b-d=4 \\ b+d=0 \end{cases} \Rightarrow a = c = 0 \quad \Rightarrow b = 2,\ d = -2$

(29)

$$\Rightarrow I = -\left[\int \frac{2\,dt}{1-t^2} - 2\int \frac{dt}{1+t^2}\right] = 2\int \frac{dt}{t^2-1} + 2\int \frac{dt}{1+t^2}$$

$$= \ln\left|\frac{t-1}{t+1}\right| + 2\arctan|t| + c \quad \text{where } t = \sqrt{\frac{1-x}{1+x}}.$$

Exercise:

Calculate the following integrals:

1) a) $I(x) = \int \frac{x\,dx}{2-x^2+2x}$

b) $J(t) = \int \frac{dt}{1+t+2\sqrt{1-t}}$

2) a) $I(t) = \int \frac{1}{(1+t)(2+t)}\,dt$ then $K(t) = \int \frac{\ln(1+t)}{(2+t)^2}\,dt$

b) $L(x) = \int \frac{e^{-x}\ln(1+e^x)}{(1+2e^{-x})^2}\,dx$

3) calculate $M(\theta) = \int \frac{d\theta}{(\cos\theta+\sin\theta)(2\cos\theta+\sin\theta)}$

Sol

1) a) $I(x) = \int \frac{x\,dx}{2-x^2+2x} = -\int \frac{x\,dx}{x^2-2x-2}$

$$= -\left[\frac{1}{2}\int \frac{(2x-2)\,dx}{x^2-2x-2} + \int \frac{dx}{x^2-2x-2}\right]$$

$$= -\frac{1}{2}\ln|x^2-2x-2| - \int \frac{dx}{x^2-2x-2}$$

$$\int \frac{dx}{x^2-2x-2} = \int \frac{dx}{(x-1)^2-3} = \int \frac{dx}{(x-1)^2-(\sqrt{3})^2} = \frac{1}{2\sqrt{3}}\ln\left|\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}\right| + c$$

$$\Rightarrow I(x) = \frac{-1}{2} \operatorname{Ln}|x^2 - 2x - 2| - \frac{1}{2\sqrt{3}} \operatorname{Ln}\left|\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}\right| + c.$$

b) $J(t) = \int \frac{dt}{1+t+2\sqrt{1-t}}$

$u = \sqrt{1-t} \Rightarrow u^2 = 1-t \Rightarrow t = 1-u^2 \Rightarrow dt = -2u\,du$

$$\Rightarrow I = \int \frac{-2u\,du}{1+1-u^2+2u} = -2\int \frac{u\,du}{2-u^2+2u} = -2\,I(u) = -2I(\sqrt{1-t})$$

2) a) $I(t) = \int \frac{dt}{(1+t)(2+t)}$ $\cancel{dt}$

$\frac{1}{(1+t)(2+t)} = \frac{a}{1+t} + \frac{b}{2+t} \Rightarrow a = 1,\ b = -1$

or $\frac{1}{(1+t)(2+t)} = \frac{2+t-t-1}{(1+t)(2+t)} = \frac{2+t-(1+t)}{(1+t)(2+t)} = \frac{1}{1+t} - \frac{1}{2+t}$

$$\Rightarrow I(t) = \int \frac{dt}{1+t} - \int \frac{dt}{2+t} = \operatorname{Ln}|1+t| - \operatorname{Ln}|2+t| + c.$$

b) $J(t) = \int \frac{\operatorname{Ln}(1+t)}{(2+t)^2}\,dt$

$u = \operatorname{Ln}(1+t) \Rightarrow u' = \frac{1}{1+t}$

$v' = \frac{1}{(2+t)^2} \Rightarrow v = \frac{-1}{(2+t)}$

$$\Rightarrow J(t) = -\frac{\operatorname{Ln}(1+t)}{2+t} + \int \frac{dt}{(1+t)(2+t)} = \frac{-\operatorname{Ln}(1+t)}{2+t} + I(t)$$

$$= \frac{-\operatorname{Ln}(1+t)}{2+t} + \operatorname{Ln}|1+t| - \operatorname{Ln}|2+t| + c$$

b) $K(x) = \int \frac{e^{-x} \ln(1+e^x)}{(1+2e^{-x})^2} dx$, $t = e^x \Rightarrow dt = e^x dx$

$$K(x) = \int \frac{\ln(1+e^x)}{e^x\left(1+\frac{2}{e^x}\right)^2} dx = \int \frac{\ln(1+e^x) . e^x dx}{(2+e^x)^2} = \int \frac{\ln(1+t) dt}{(2+t)^2}$$

$$= J(t) = J(e^x).$$

c) $M(\theta) = \int \frac{d\theta}{(\cos\theta + \sin\theta)(2\cos\theta + \sin\theta)}$

$$A(\theta) = \frac{d\theta}{(\cos\theta + \sin\theta)(2\cos\theta + \sin\theta)} \Rightarrow A(\pi + \theta) = A(\theta)$$

$t = \tan\theta \Rightarrow dt = \frac{1}{\cos^2\theta} d\theta$

$$M(\theta) = \int \frac{d\theta}{(1+\tan\theta)(2+\tan\theta)\cos^2\theta} = \int \frac{dt}{(1+t)(2+t)} = I(t) = I(\tan\theta).$$

integrals of the form $I = \int f(x, \sqrt{ax^2+bx+c}) dx$

In this section, we study the integrals of the form $I = \int f(x, y) dx$ where $y = \sqrt{ax^2+bx+c}$ and f is a rational function in x and y.

the term ax^2+bx+c can be transformed into one of the following forms:

1) $\alpha(k^2-u^2)$ $(\alpha>0)$: we make the change of variable

$\cancel{u=ksht}$. $u=k\sin t$.

2) $\alpha(k^2+u^2)$ $(\alpha>0)$: we make the change of variables

$u=k\,sh\,t$ or $u=k\tan t$.

3) $\alpha(k^2-u^2)$ $(\alpha>0)$: we make the change of variables

$u=k\,ch\,t$ or $u=\dfrac{k}{\cos t}$.

Example: calculate $I=\int\sqrt{-x^2+2x+8}\,dx$

Sol: $-x^2+2x+8=-[x^2-2x-8]=-[(x-1)^2-9]=[3^2-(x-1)^2]$

$\Rightarrow I=\int\sqrt{3^2-(x-1)^2}\,dx$

Let $x-1=3\sin t \Rightarrow x=1+3\sin t \Rightarrow dx=3\cos t\,dt$

$\Rightarrow I=\int\sqrt{3^2-3^2\sin^2 t}\cdot 3\cos t\,dt=9\int\sqrt{1-\sin^2 t}\cdot\cos t\,dt$

$=9\int\cos^2 t\,dt=\frac{9}{2}\int(1+\cos 2t)\,dt=\frac{9}{2}t+\frac{9}{4}\sin 2t+c$

$=\frac{9}{2}t+\frac{9}{2}\cos t\cdot\sin t+c=\frac{9}{2}t+\frac{9}{2}\sqrt{1-\sin^2 t}\cdot\sin t+c$

$x-1=3\sin t \Rightarrow \sin t=\frac{x-1}{3} \Rightarrow t=\arcsin\left(\frac{x-1}{3}\right)$

$\Rightarrow I = \frac{9}{2} \arcsin\left(\frac{x-1}{3}\right) + \frac{9}{2}\sqrt{1-\left(\frac{x-1}{3}\right)^2} \cdot \left(\frac{x-1}{3}\right) + C.$

particular case: $I = \int \frac{dx}{\sqrt{ax^2+bx+c}}$

① $I = \int \frac{dx}{\sqrt{2x^2+4x+6}}$

$2x^2+4x+6 = 2\left[x^2+2x+3\right] = 2\left[(x+1)^2-1+3\right] = 2\left[(x+1)^2+2\right]$

$\Rightarrow I = \int \frac{dx}{\sqrt{2\left((x+1)^2+2\right)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+1)^2+2}} = \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2+2}}$

$= \frac{1}{\sqrt{2}} \ln\left|t+\sqrt{t^2+2}\right| + C.$

$\Rightarrow I = \frac{1}{\sqrt{2}} \ln\left|(x+1)+\sqrt{(x+1)^2+2}\right| + C.$

Remark: $\int \frac{dt}{\sqrt{t^2+k}} = \ln\left|t+\sqrt{t^2+k}\right| + C.$

2) $I = \int \frac{dx}{\sqrt{-4x^2+4x+3}}$

$-4x^2+4x+3 = -4\left[x^2-x-\frac{3}{4}\right] = -4\left[\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-\frac{3}{4}\right]$

$= -4\left[\left(x-\frac{1}{2}\right)^2-1\right] = 4\left[1-\left(x-\frac{1}{2}\right)^2\right]$

$I = \int \frac{dx}{\sqrt{4\left[1-\left(x-\frac{1}{2}\right)^2\right]}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2+1}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(x-\frac{1}{2}\right)^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}}$

$$\Rightarrow I = \frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}} \quad \text{where } t = (x-\tfrac{1}{2})$$

$$= \frac{1}{2}\arcsin(t) + c = \frac{1}{2}\arcsin\left(x-\tfrac{1}{2}\right) + C.$$

Remark: $\displaystyle\int \frac{dt}{\sqrt{a^2-t^2}} = \arcsin\left(\frac{t}{a}\right) + c.$

. particular case 2: $\displaystyle I = \int \frac{\alpha x + \beta}{\sqrt{ax^2+bx+c}}\,dx$

Ex

$$I = \int \frac{(2x+1)\,dx}{\sqrt{-2x^2+4x+8}}.$$

$$I = \frac{-1}{2}\int \frac{(-4x+4)\,dx}{\sqrt{-2x^2+4x+8}} + 3\int \frac{dx}{\sqrt{-2x^2+4x+8}}$$

$$\int \frac{(-4x+4)\,dx}{\sqrt{-2x^2+4x+8}} = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= 2\sqrt{t} + c = 2\sqrt{-2x^2+4x+8} + C.$$

$$\int \frac{dx}{\sqrt{-2x^2+4x+8}} = \int \frac{dx}{\sqrt{-2(x^2-2x-4)}} = \int \frac{dx}{\sqrt{-2[(x-1)^2-5]}}$$

$$= \int \frac{dx}{\sqrt{2(5-(x-1)^2)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(\sqrt{5})^2-(x-1)^2}} = \frac{1}{\sqrt{2}}\cdot\arcsin\left(\frac{x-1}{\sqrt{5}}\right) + C$$

$\Rightarrow I(x) = -\sqrt{-2x^2+4x+8} + \frac{3}{\sqrt{2}} \arcsin\left(\frac{x-1}{\sqrt{5}}\right) + c.$

Integrals of the form:

1) $\int f(x, (a^2+x^2)^\alpha)\,dx$: we make the change of variables
$x = a\tan t$.

2) $\int f(x, (a^2-x^2)^\alpha)\,dx$: we make the change of variables
$x = a\sin t$.

Example: $I = \int \frac{dx}{(1+x^2)^2}$

$x = \tan t \Rightarrow dx = (1+\tan^2 t)\,dt$

$$I = \int \frac{(1+\tan^2 t)\,dt}{(1+\tan^2 t)^2} = \int \frac{dt}{(1+\tan^2 t)} = \int \cos^2 t\,dt = \int \frac{1+\cos 2t}{2}\,dt$$

$$= \frac{1}{2} t + \frac{1}{4}\sin 2t + c = \frac{1}{2} t + \frac{1}{2}\,\frac{\tan t}{1+\tan^2 t} + c$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\cdot\frac{x}{1+x^2} + c$$

$(x = \tan t \Rightarrow t = \arctan x)$

Definite integral

$$I = \int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$ where G is a primitive of $f(x)$.

$$\int_\alpha^\beta f(\varphi(x))\cdot\varphi'(x)\,dx = \int_a^b f(t)\,dt$$ where $t = \varphi(x)$
$a = \varphi(\alpha)$, $b = \varphi(\beta)$.

Integration by parts:

$$\int_a^b u v'\,dx = [uv]_a^b - \int_a^b u'v\,dx$$

Exercise: Let $J = \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \frac{u^2}{1-u^2}\,du.$

1) a) Determine a, b and c such that:

$$\frac{u^2}{1-u^2} = \frac{a}{1-u} + \frac{b}{1+u} + c$$

b) Determine the value of J.

2) a) calculate $H = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos x}\,dx$

b) $K = \displaystyle\int_0^{\pi/4} \cos x \cdot \ln(\cos x)\,dx$

Sol:

1) a) $\frac{u^2}{1-u^2} = \frac{a}{1-u} + \frac{b}{1+u} + c$

$\Rightarrow a(1+u) + b(1-u) + c(1-u^2) = u^2$

$\Rightarrow a + au + b - bu + c - cu^2 = u^2 \Rightarrow (a+b+c) + (a-b)u - cu^2 = u^2$

$\Rightarrow \begin{cases} a+b+c=0 \\ a-b=0 \\ c=-1 \end{cases} \Rightarrow \begin{cases} a+b=1 \\ a-b=0 \end{cases} \Rightarrow a = \frac{1}{2},\ b = \frac{1}{2}.$

$\Rightarrow \frac{u^2}{1-u^2} = \frac{1}{2} \cdot \frac{1}{1-u} + \frac{1}{2} \cdot \frac{1}{1+u} - 1$

b) $J = \int_0^{\frac{\sqrt{2}}{2}} \frac{u^2}{1-u^2} du = \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} \frac{du}{1-u} + \frac{1}{2} \int_0^{\frac{\sqrt{2}}{2}} \frac{du}{1+u} - \int_0^{\frac{\sqrt{2}}{2}} du$

$= -\frac{1}{2} \left[\ln|1-u| \right]_0^{\frac{\sqrt{2}}{2}} + \frac{1}{2} \left[\ln|1+u| \right]_0^{\frac{\sqrt{2}}{2}} - \left(\frac{\sqrt{2}}{2} - 0 \right)$

$= -\frac{1}{2} \ln\left(1 - \frac{\sqrt{2}}{2}\right) + \frac{1}{2} \ln\left(1 + \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}.$

2) $M = \int_0^{\pi/4} \frac{\sin^2 x}{\cos x} dx$

$A(x) = \frac{\sin^2 x}{\cos x} dx$

$A(\pi - x) = A(x) \Rightarrow t = \sin x \Rightarrow dt = \cos x \, dx$

For $x=0 \Rightarrow t=\sin 0=0$

For $x=\frac{\pi}{4} \Rightarrow t=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$$H=\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos x}\,dx=\int_0^{\frac{\pi}{4}} \frac{\sin^2 x\cdot\cos x\,dx}{\cos^2 x}=\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{1-\sin^2 x}\cdot\cos x\,dx$$

$$=\int_0^{\frac{\sqrt{2}}{2}} \frac{t^2}{1-t^2}\,dt=J.$$

b) $K=\int_0^{\pi/4} \cos x\cdot\ln(\cos x)\,dx$

$u=\ln(\cos x)\Rightarrow u'=\frac{-\sin x}{\cos x}$

$v'=\cos x\Rightarrow v=\sin x$

$$\Rightarrow K=\left[\sin x\ln(\cos x)\right]_0^{\pi/4}+\int_0^{\pi/4}\frac{\sin^2 x}{\cos x}\,dx$$

$$=\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2}}{2}\right)+H.$$

Ex: calculate the following integrals.

1) $I(x)=\int \frac{(x+1)}{\sqrt{-4x^2+4x+3}}\,dx$

2) $J(x)=\int \frac{2-\tan x}{(\cos x+2\sin x)\cos x}\,dx$

3) $K(x)=\int \frac{dx}{(2+\cos x)\tan(\frac{x}{2})}$

Sol:

1) $I(x) = \int \frac{x+1}{\sqrt{-4x^2+4x+3}}\,dx = \frac{-1}{8}\int \frac{(-8x+4)\,dx}{\sqrt{-4x^2+4x+3}} + \frac{3}{2}\int \frac{dx}{\sqrt{-4x^2+4x+3}}$

$\int \frac{(-8x+4)\,dx}{\sqrt{-4x^2+4x+3}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{-4x^2+4x+3} + C$

$\int \frac{dx}{\sqrt{-4x^2+4x+3}} = \int \frac{dx}{\sqrt{-4(x^2-x-3/4)}} = \int \frac{dx}{\sqrt{-4[(x-\frac{1}{2})^2-1]}}$

$= \int \frac{dx}{\sqrt{4(1-(x-\frac{1}{2})^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1^2-(x-\frac{1}{2})^2}} = \frac{1}{2}\arcsin(x-\frac{1}{2}) + C$

$\Rightarrow I = \frac{-1}{4}\sqrt{-4x^2+4x+3} + \frac{3}{4}\arcsin(x-\frac{1}{2}) + C$

2) $J(x) = \int \frac{2-\tan x}{(\cos x + 2\sin x)\cos x}\,dx$

$t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x}\,dx$

$J(x) = \int \frac{2-\tan x}{(1+\tan x)\cdot\cos^2 x}\,dx = \int \frac{2-t}{1+t}\,dt$

$$\begin{array}{r|l} -t+2 & t+1 \\ \hline -t-1 & -1 \\ \hline 3 & \end{array}$$

$\Rightarrow (-t+2) = -(t+1) + 3$

$\Rightarrow \frac{2-t}{t+1} = -1 + \frac{3}{1+t}$

$\Rightarrow J(x) = \int \left(-1 + \frac{3}{1+t}\right) dt = -t + 3\ln|1+t| + c$

$\Rightarrow J(x) = -\tan x + 3\ln|1+\tan x| + c.$

3) $K(x) = \int \frac{dx}{(2+\cos x).\tan x/2}$, $t = \tan\frac{x}{2} \Rightarrow dx = \frac{2dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$\Rightarrow k = \int \frac{\frac{2dt}{1+t^2}}{\left(2 + \frac{1-t^2}{1+t^2}\right).t} = \int \frac{2dt}{t(3+t^2)}$

$= 2\int \frac{dt}{(3+t^2)t} = \frac{2}{3}\int \frac{(3+t^2-t^2)}{(3+t^2)t} dt$

$= \frac{2}{3}\int \left(1 - \frac{t}{3+t^2}\right) dt = \frac{2}{3}t - \frac{1}{3}\int \frac{2t\,dt}{3+t^2}$

$= \frac{2}{3}t - \frac{1}{3}\ln(3+t^2) + c$ where $t = \tan(x/2)$.

Exercise:

1) a) Verify that: $\frac{t^3}{(2+t)^2} = t - 4 + \frac{12}{2+t} - \frac{8}{(2+t)^2}$

b) calculate $I(x) = \int \frac{\tan^3 x}{(2\cos x + \sin x)^2} dx$

2) Calculate $J(x) = \int \frac{dx}{5+3\cos x}$ and $K(x) = \int \frac{dx}{3-5\cos x}$

3) Calculate $L(x) = \int \frac{1}{x}\cdot\sqrt{\frac{x}{x+1}}\, dx$.

a) easy

b) $I(x) = \int \frac{\tan^3 x}{(2\cos x + \sin x)^2} dx$

$A(x) = \frac{\tan^3 x}{(2\cos x + \sin x)^2} \Rightarrow A(-x) = A(x) \Rightarrow t = \tan x$

$t = \tan x \Rightarrow dt = \frac{1}{\cos^2 x} dx$

$I(x) = \int \frac{\tan^3 x}{(2 + \tan x)^2 \cdot \cos^2 x} dx = \int \frac{t^3}{(2+t)^2} dt$

$= \int (t-4)\, dt + \int \frac{12}{t+2} dt - \int \frac{8}{(2+t)^2} dt$

$= \frac{t^2}{2} - 4t + 12 \ln|t+2| + \frac{8}{2+t} + c.$

2°) $J(x) = \int \frac{dx}{5 + 3\cos x}$

$t = \tan x/2 \Rightarrow dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$\Rightarrow J = \int \frac{\frac{2dt}{1+t^2}}{5 + 3\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{dt}{4+t^2} = \int \frac{dt}{2^2+t^2} = \frac{1}{2} \arctan(t/2) + c$

where $t = \tan(x/2)$.

$$K(x) = \int \frac{dx}{3-5\cos x}$$

$t = \tan x/2 \Rightarrow dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$$\Rightarrow K = \int \frac{\frac{2dt}{1+t^2}}{3-5\left(\frac{1-t^2}{1+t^2}\right)} = -\int \frac{dt}{1-4t^2} = \int \frac{dt}{4t^2-1}$$

$$= \frac{1}{2}\int \frac{(2dt)}{(2t)^2-1} = \frac{1}{2} \cdot \frac{1}{2} \ln \left| \frac{2t-1}{2t+1} \right|$$

$$= \frac{1}{4} \ln \left| \frac{2t-1}{2t+1} \right| + C.$$

where $t = \tan(x/2)$

$$\Rightarrow L(x) = \int \frac{1}{x} \cdot \sqrt{\frac{x}{x+1}}\, dx$$

$t = \sqrt{\frac{x}{1+x}} \Rightarrow t^2 = \frac{x}{1+x} \Rightarrow x = t^2 + xt^2$

$\Rightarrow x(1-t^2) = t^2 \Rightarrow$

$$x = \frac{t^2}{1-t^2}$$

$$dx = \left(\frac{t^2}{1-t^2}\right)' dt = \frac{2t(1-t^2)+2t(t^2)}{(1-t^2)^2} dt = \frac{2t\,dt}{(1-t^2)^2}$$

$$\Rightarrow L(x) = \int \frac{1-t^2}{t^2} \cdot t \cdot \frac{2t\,dt}{(1+t^2)^2} = 2\int \frac{dt}{1-t^2}$$

$$= -2\int \frac{dt}{t^2-1} = -\ln\left|\frac{t-1}{t+1}\right| + c$$