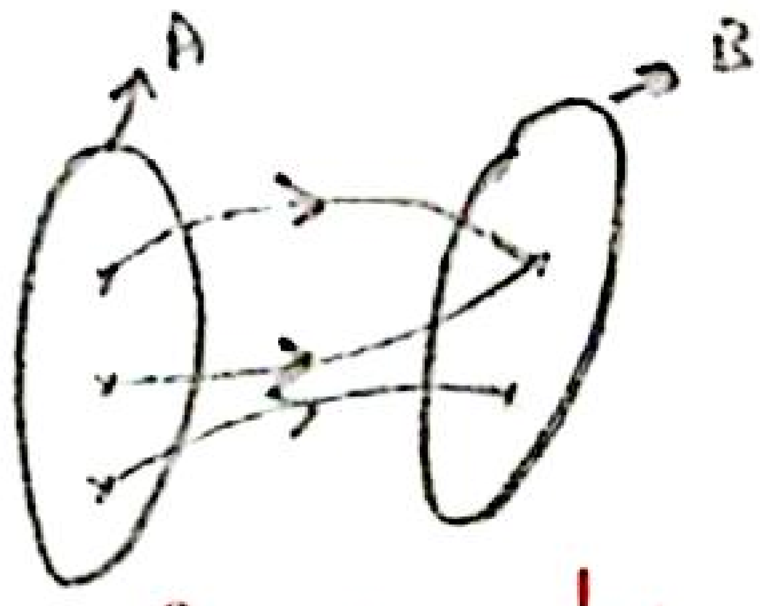
Usual Functions

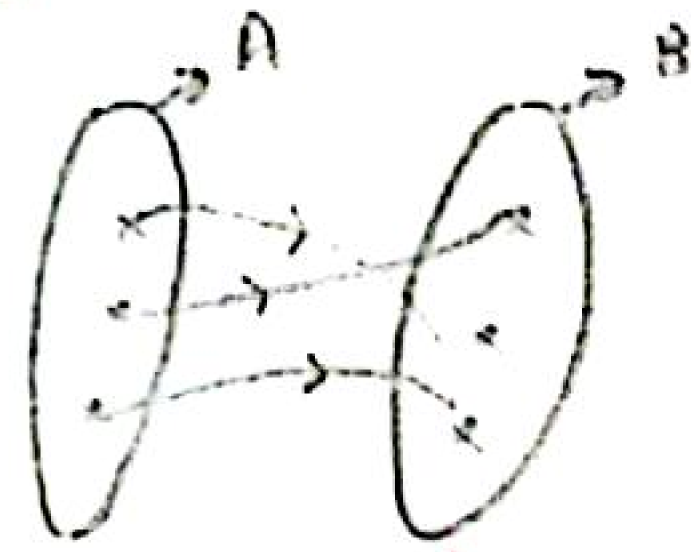
Inverse function:

Def: 1. we say that f is injective iff whenever $f(x)=f(y) \Rightarrow x=y$

i.e $f(x) \neq f(y)\ \forall x \neq y$.



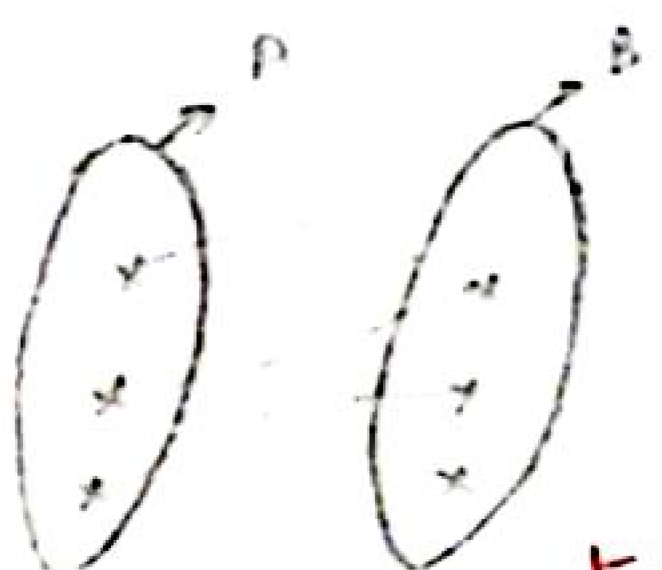
f is not injective.



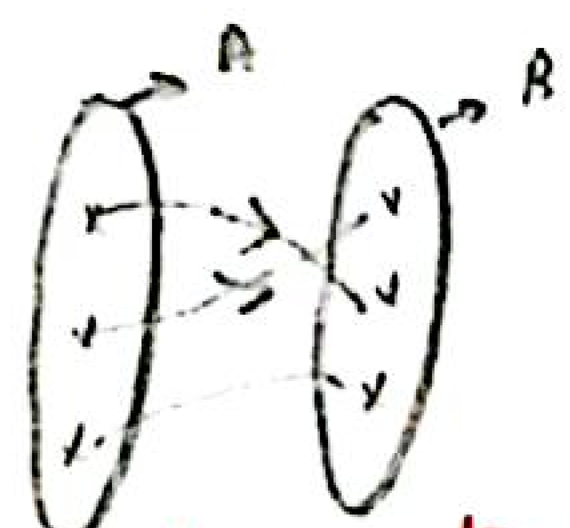
f is injective.

2. we say that f is surjective iff $\forall y \in B, \exists x \in A /$

$$y=f(x) \quad (f: A \longrightarrow B).$$



f is not surjective



f is surjective.

3. we say that f is bijective iff f is injective and surjective.

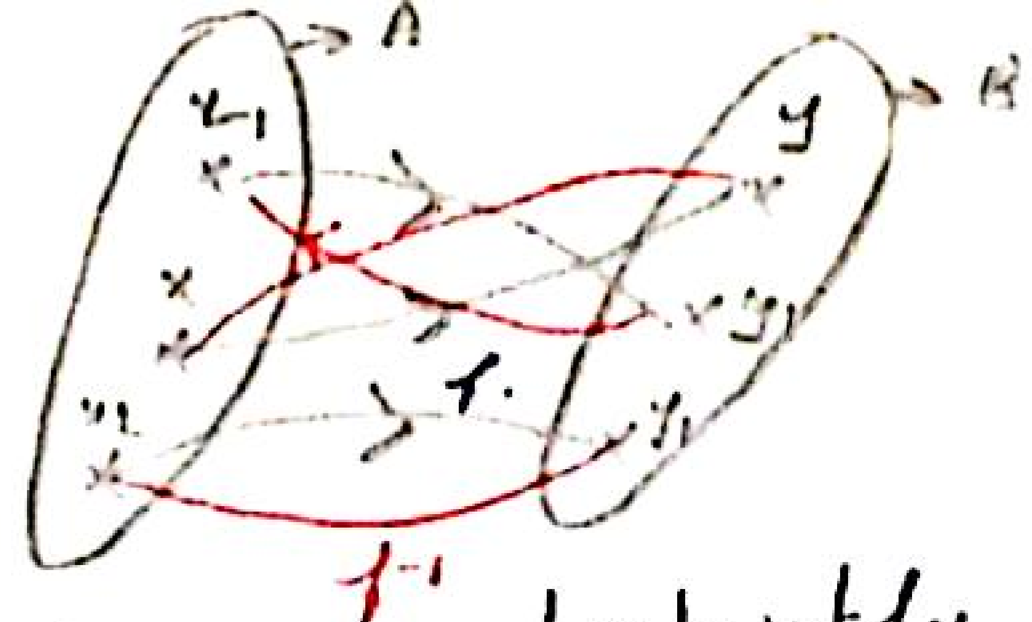
If f is bijective then f^{-1} exists and if:

$$f: A \longrightarrow B \text{ then}$$

$$f^{-1}: B \longrightarrow A \text{ and } f \circ f^{-1} = f^{-1} \circ f = Id.$$

if $x \in A \Rightarrow f^{-1} \circ f(x) = x$.

if $y \in B \Rightarrow f \circ f^{-1}(y) = y$



Theorem: If $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and strictly monotone function then f is a bijective from I into $J = f(I)$. Moreover, f admits an inverse function f^{-1} s.t $f^{-1} : J \longrightarrow I$ and:

1) f^{-1} is continuous on J.

2) f^{-1} is strictly monotone with the same sense of monotonicity of f.

3) $x \in I \Rightarrow y = f(x) \in J$ and $y = f(x), x \in I \Leftrightarrow x = f^{-1}(y), y \in J$.

4) $f \circ f^{-1}(y) = y \ \forall y \in J$, $f^{-1} \circ f(x) = x \ \forall x \in I$.

5) the graphs of f and f^{-1} are symmetric with respect to $y = x$.

theorem. Let $f : I \longrightarrow J$ be a bijective function. If f is differentiable at $a \in I$ with $f'(a) \neq 0$, then f^{-1} is differentiable at $b = f(a)$ with

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

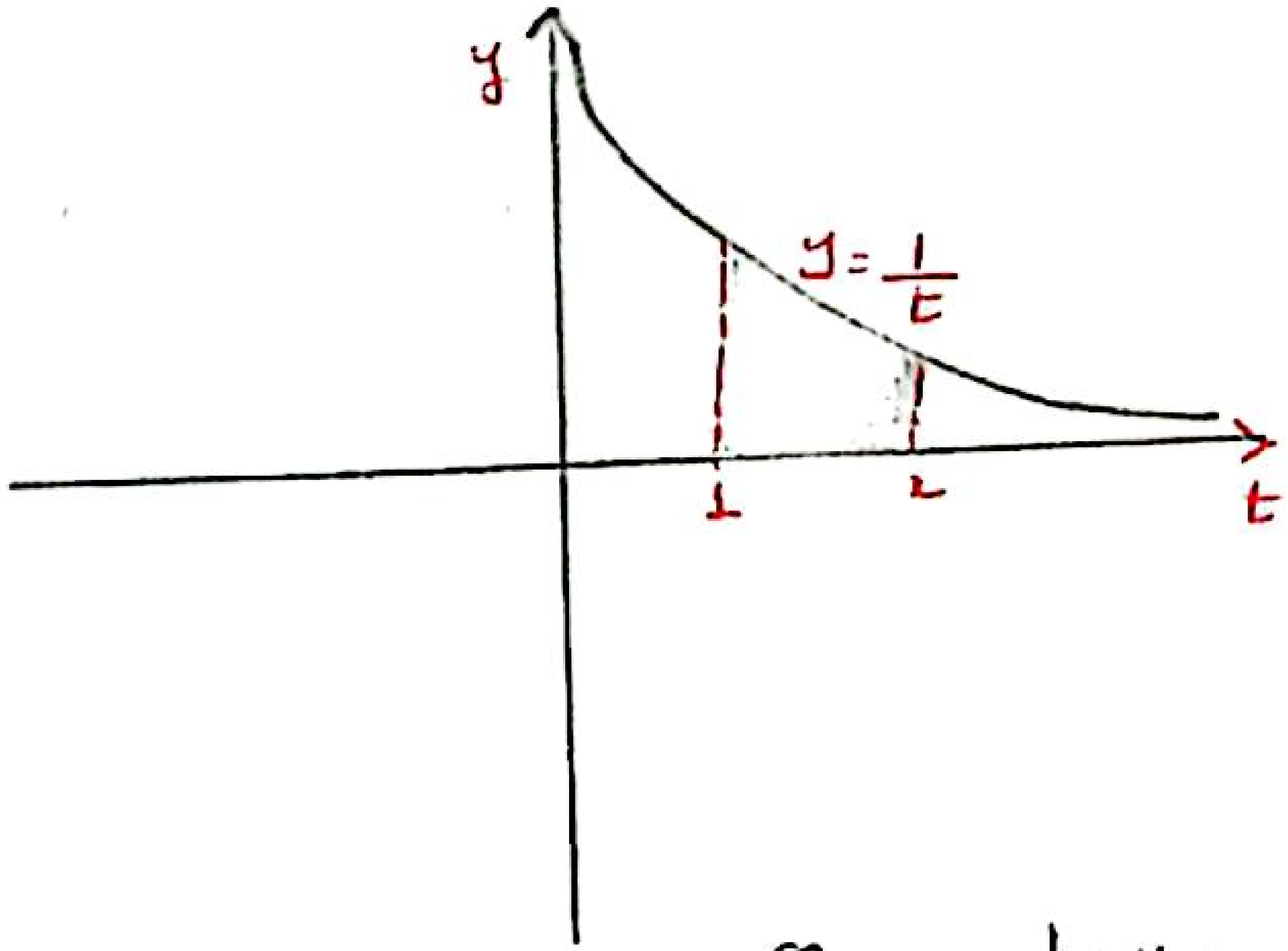
Usual Functions

1. Logarithm Function Ln

Def: We call Logarithm function denoted by Ln, the primitive on $\mathbb{R}_+^*$ of the function $\frac{1}{x}$ with $Ln\,1 = 0$.

$$Ln :]0, +\infty[\longrightarrow \mathbb{R}$$
$$x \longmapsto Ln\,x \;/\; Ln'\,x = \frac{1}{x} \text{ and } Ln\,1 = 0.$$

$$Ln\,x = \int_1^x \frac{dt}{t}$$



properties: Let $x, y \in]0, +\infty[$ and $\alpha \in \mathbb{Q}$. we have:

1) $Ln\,xy = Ln\,x + Ln\,y$

2) $Ln\,\frac{x}{y} = Ln\,x - Ln\,y$

3) $Ln\,x^{\alpha} = \alpha\,Ln\,x$

4) $Ln\,\frac{1}{x} = -Ln\,x$

proposition:

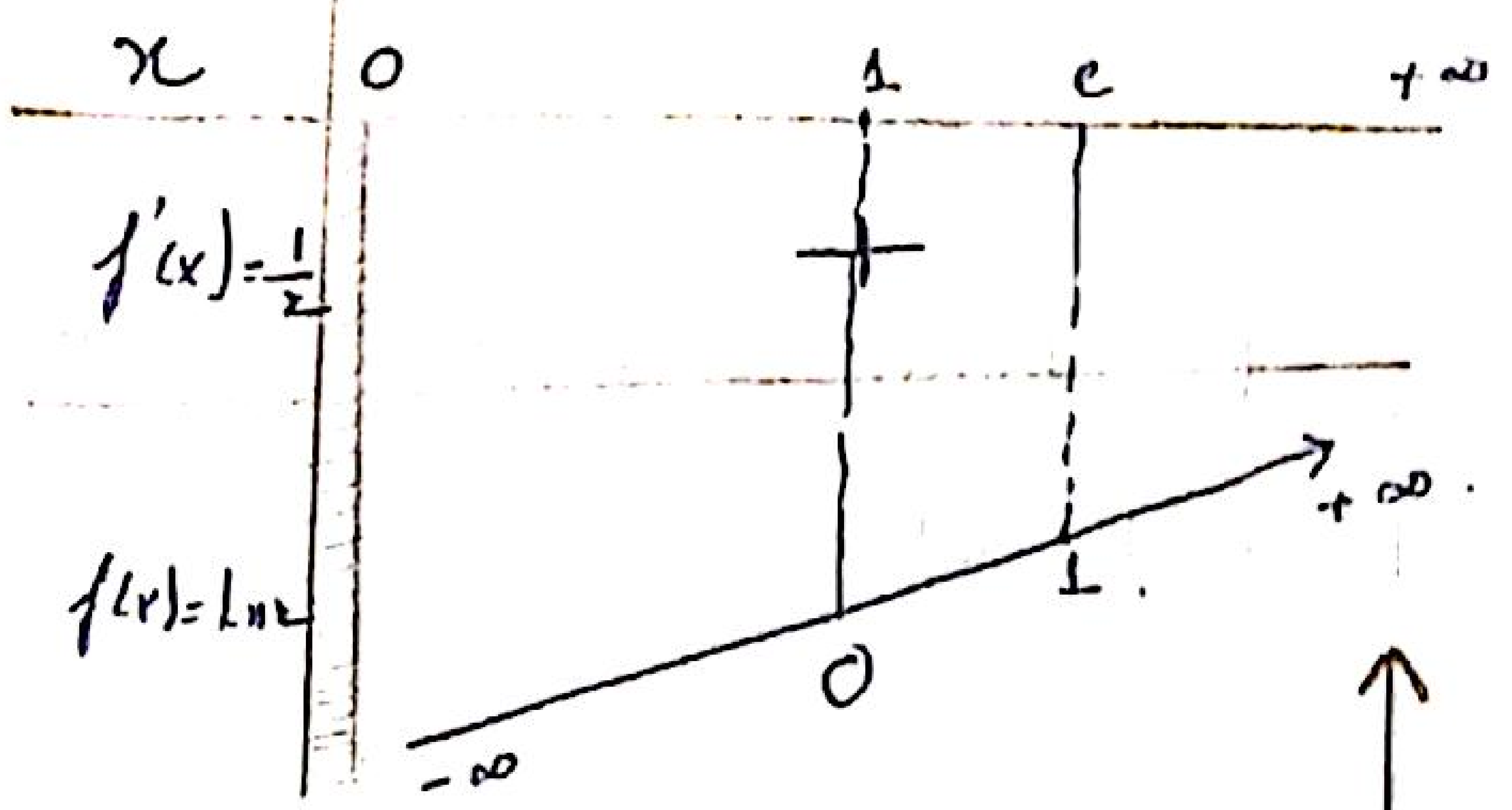
1) $\lim\limits_{x \to +\infty} Ln\,x = +\infty$

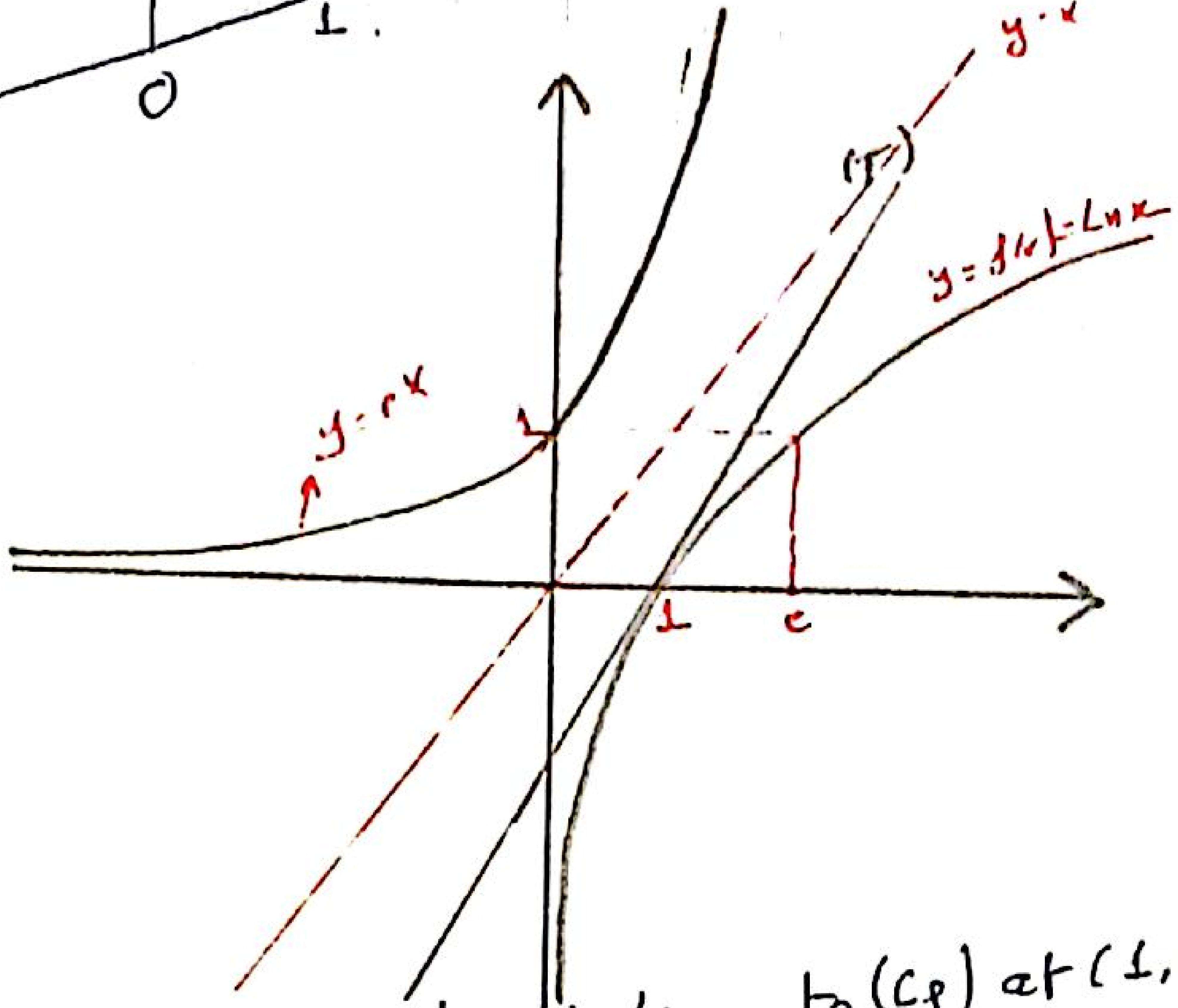
2) $\lim\limits_{x \to 0^+} Ln\,x = -\infty.$

3) $\lim\limits_{x \to +\infty} \frac{Ln\,x}{x} = 0$

4) $\lim\limits_{x \to 0^+} x\,Ln\,x = 0$

5) $\lim\limits_{x \to 0} \frac{Ln(1+x)}{x} = 1.$





Let (T) be the equation of the tangent line to (Cf) at (1, 0)

$$(T)\quad y = f'(1)(x-1) + f(1)$$

$$(T):\ y = x - 1$$

so $\ln x \leq x - 1 \quad \forall x \in]0, +\infty[$.

Exponential function

the function $f(x) = \ln x$ is continuous and strictly increasing on $]0, +\infty[$ with image $\mathbb{R}$, then it admits an inverse function called the exponential, denoted by e^x or exp such that

$$f :]0,+\infty[\longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x) = \ln x$$

$$f^{-1} : \mathbb{R} \longrightarrow]0,+\infty[$$
$$y \longrightarrow f^{-1}(y) = e^y .$$

$y = \ln x,\ x \in]0,+\infty[\Leftrightarrow x = e^y,\ y \in \mathbb{R} \quad (y = f(x) \Leftrightarrow x = f^{-1}(y)).$

. $(f \circ f^{-1})(y) = y \ \forall y \in \mathbb{R} \Leftrightarrow f(f^{-1}(y)) = y \ \forall y \in \mathbb{R}$

$\Leftrightarrow \ln e^y = y \ \forall y \in \mathbb{R}.$

. $(f^{-1} \circ f)(x) = x \ \forall x \in]0,+\infty[\Leftrightarrow f^{-1}(f(x)) = x \ \forall x \in]0,+\infty[$

$\Leftrightarrow e^{\ln x} = x \ \forall x \in]0,+\infty[$

. $e^y > 0 \ \forall y \in \mathbb{R}.$

propriétés

1) $e^{a+b} = e^a . e^b$

2) $\dfrac{e^a}{e^b} = e^{a-b}$

3) $e^{-x} = \dfrac{1}{e^x}$

4) $(e^x)^\alpha = e^{\alpha x} \ \forall \alpha \in \mathbb{R}$

5) $\lim\limits_{x \to +\infty} e^x = +\infty$

6) $\lim\limits_{x \to +\infty} \dfrac{e^x}{x} = +\infty$

7) $\lim\limits_{x \to -\infty} e^x = 0$

8) $\lim\limits_{x \to +\infty} \dfrac{e^x}{x^\alpha} = 0 \ \forall \alpha > 0$

9) $\lim\limits_{x \to -\infty} |x|^\alpha e^x = 0 \ \forall \alpha > 0$

Derivative: $(e^x)' = e^x \ \forall x \in \mathbb{R}$.

$f:]0,+\infty[\longleftrightarrow \mathbb{R}$ $\qquad$ $f^{-1}: \mathbb{R} \longrightarrow]0,+\infty[$

$x \longrightarrow f(x) = \ln x$ $\qquad$ $y \longrightarrow f^{-1}(y) = e^y$.

Let $b = f(a) = \ln a \Rightarrow f'(a) = \frac{1}{a} \neq 0 \ \forall a \in]0,+\infty[$

So $(f^{-1})'(b) = \dfrac{1}{f'(a)} = \dfrac{1}{\left(\frac{1}{a}\right)} = a = e^b$. $(b = \ln a \Leftrightarrow a = e^b)$.

So $(e^x)' = e^x \ \forall x \in \mathbb{R}$.

Exercise: Let g be the function defined over $\mathbb{R}$ by:

$$g(t) = t + \ln\left(\frac{e^{-t}+1}{2}\right)$$

1) Determine it's derivative g'.

2) Show that g' is strictly increasing

3) Using the Mean Value theorem, show that:

$$\forall x > 0, \ \frac{x}{2} \le g(x) \le \frac{x e^x}{1+e^x}$$

Sol

1) $g'(t) = 1 + \dfrac{\left(\frac{-e^{-t}}{2}\right)}{\left(\frac{e^{-t}+1}{2}\right)} = 1 - \dfrac{e^{-t}}{e^{-t}+1} = \dfrac{1}{e^{-t}+1} = \dfrac{e^t}{1+e^t}$.

2) $g''(t) = \dfrac{e^t(1+e^t) - e^t(e^t)}{(1+e^t)^2} = \dfrac{e^t}{(1+e^t)^2} > 0 \quad \forall t \in \mathbb{R}$

So g' is strictly increasing.

3) Apply the M.V.T to $g(t)$, $t \in [0,x]$ we have:

- g is continuous on $[0,x]$, $x > 0$.
- g is differentiable on $]0,x[$

So by M.V.T, there exists at least $c \in]0,x[$ s.t

$$g(x) - g(0) = (x-0) \cdot g'(c).$$

$\Rightarrow g(x) = x\, g'(c)$, $g(0) = 0$.

$0 < c < x$ and g' strictly increasing $\Rightarrow g'(0) < g'(c) < g'(x)$

$\Rightarrow \dfrac{1}{2} < g'(c) < \dfrac{e^x}{1+e^x}$

$\Rightarrow \dfrac{x}{2} < x\, g'(c) < \dfrac{x e^x}{1+e^x}$

then $\dfrac{x}{2} < g(x) < \dfrac{x e^x}{1+e^x} \quad \forall x > 0$

. The inverse trigonometric functions

1. The arc sine function.

The function $f(x) = \sin x$ is continuous and strictly increasing over $[-\pi/2, +\pi/2]$ ($f'(x) = \cos x > 0 \; \forall x \in]-\pi/2, +\pi/2[$) with image $[-1, +1]$ so it admits an inverse function called the arc sine function denoted by arcsin such that:

$$f : [-\pi/2 ; \tfrac{\pi}{2}] \longrightarrow [-1, 1]$$
$$x \longmapsto f(x) = \sin x.$$

$$f^{-1} : [-1, +1] \longrightarrow [-\pi/2, +\pi/2]$$
$$y \longrightarrow f^{-1}(y) = \arcsin y.$$

i) $y = f(x), x \in [-\pi/2, +\pi/2] \Leftrightarrow x = f^{-1}(y), y \in [-1, +1]$

$\Leftrightarrow y = \sin x, \; x \in [-\pi/2, +\pi/2] \Leftrightarrow x = \arcsin y, \; y \in [-1, +1]$.

ii) $(f \circ f^{-1})(y) = y \Leftrightarrow f(f^{-1}(y)) = y \quad \forall y \in [-1, +1]$

$\Leftrightarrow \sin(\arcsin y) = y \; \forall y \in [-1, +1]$

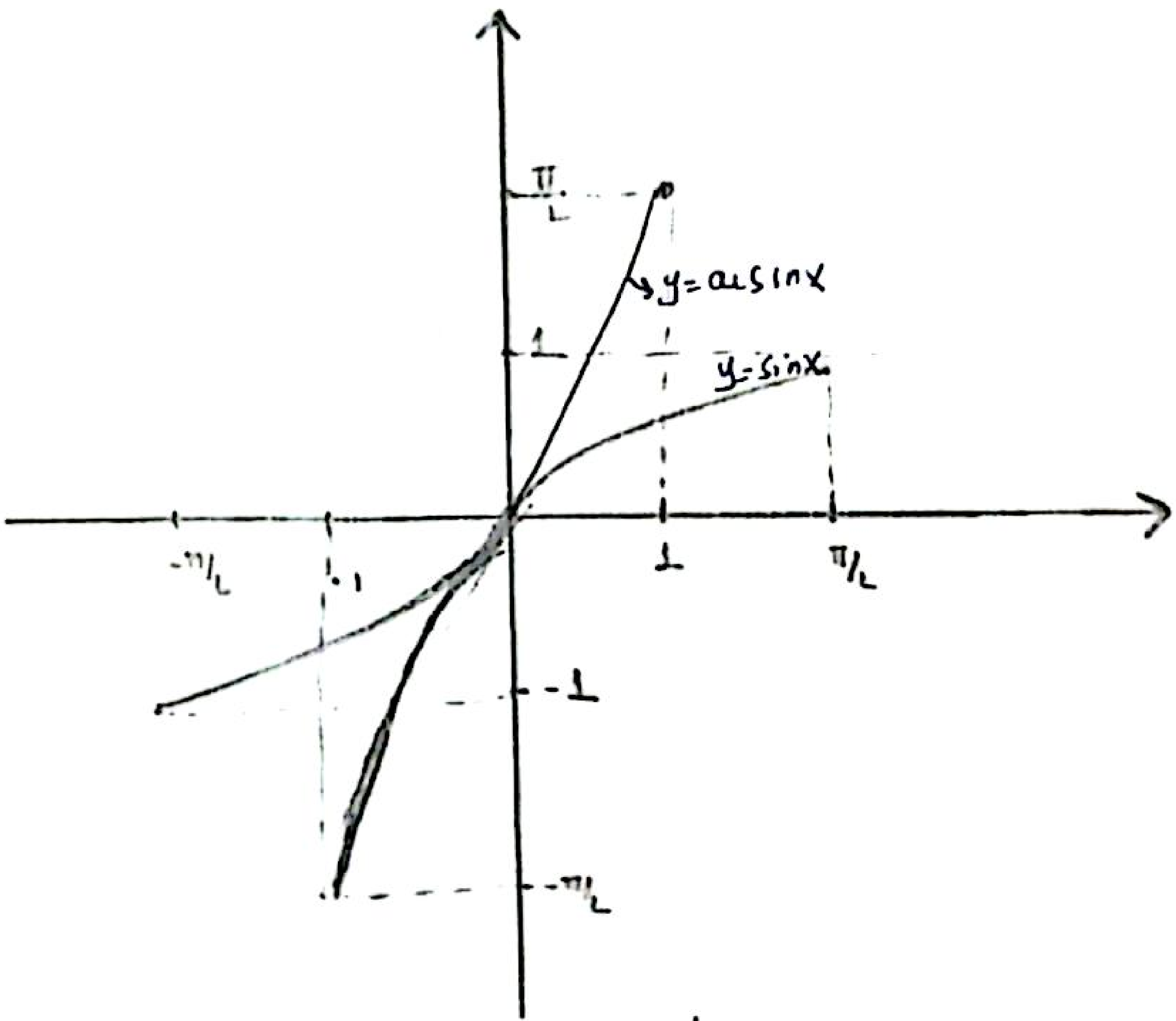
iii) $(f^{-1} \circ f)(x) = x \quad \forall x \in [-\pi/2, +\pi/2] \Leftrightarrow \arcsin(\sin x) = x$

$\forall x \in [-\pi/2, +\pi/2]$

• the function arcsin is continuous, strictly increasing and odd on $[-1,+1]$.

Some particular values.

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\arcsin x$	0	$\frac{\pi}{6}$	$\pi/4$	$\pi/3$	$\pi/2$



Derivative $\forall x \in]-1,+1[$, $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$

Let $b = f(a) = \sin a$, $a \in [-\pi/2, +\pi/2]$

$f'(a) = \cos a$ so $f'(a) = 0 \Leftrightarrow a = \pm\pi/2$

$\Rightarrow b \neq f(\pm\pi/2) \neq \pm 1$.

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{\cos a} = \frac{1}{\sqrt{1-\sin^2 a}} = \frac{1}{\sqrt{1-b^2}} \cdot \forall b \in]-1,+1[.$$

Remark: $\arcsin(u(x))' = \dfrac{u'(x)}{\sqrt{1-u^2(x)}}$.

• The arc Cosine function.

The function $f(x) = \cos x$ is continuous and strictly decreasing on $[0,\pi]$ ($f'(x) = -\sin x < 0$ $\forall x \in]0,\pi[$) with image $[-1,+1]$ so it admits an inverse function called the arc cosine function denoted by arccos s.t:

$$f : [0,\pi] \longrightarrow [-1,+1]$$
$$x \longrightarrow f(x) = \cos x$$

$$f^{-1} : [-1,+1] \longrightarrow [0,\pi]$$
$$y \longrightarrow f^{-1}(y) = \cos y.$$

i) $y = \cos x$, $x \in [0,\pi] \Leftrightarrow x = \arccos y$, $y \in [-1,+1]$.

ii) $\arccos(\cos x) = x$ $\forall x \in [0,\pi]$

iii) $\cos(\arccos y) = y$ $\forall y \in [-1,+1]$

The function arc cosine is continuous and strictly decreasing on $[-1,+1]$.

Some particular values:

x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
arccos x	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$	0

Derivative: $\forall x \in]-1,+1[$, $\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$

$$\arccos(u(x))' = \frac{-u'(x)}{\sqrt{1-u^2(x)}}.$$

3. the arc tangent function.

the function $f(x) = \tan x$ is continuous and strictly increasing on $]-\pi/2, +\pi/2[$ $(f'(x) = 1 + \tan^2(x) > 0 \ \forall x \in]-\pi/2, +\pi/2[)$ with image $\mathbb{R}$. So, it admits an inverse function called the arc tangent function denoted by arc tan such that:

$$f :]-\pi/2, +\pi/2[\longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x) = \tan x$$

$$f^{-1} : \mathbb{R} \longrightarrow]-\pi/2, +\pi/2[$$
$$y \longrightarrow f^{-1}(y) = \arctan y$$

i) $y = \tan x$, $x \in]-\pi/2, +\pi/2[\Leftrightarrow x = \arctan y$, $y \in \mathbb{R}$.

ii) $\arctan(\tan x) = x \quad \forall x \in]-\pi/2, +\pi/2[$.

iii) $\tan(\arctan y) = y \quad \forall y \in \mathbb{R}$.

. The function arctan is continuous, strictly increasing and odd on $\mathbb{R}$.

Some particular values:

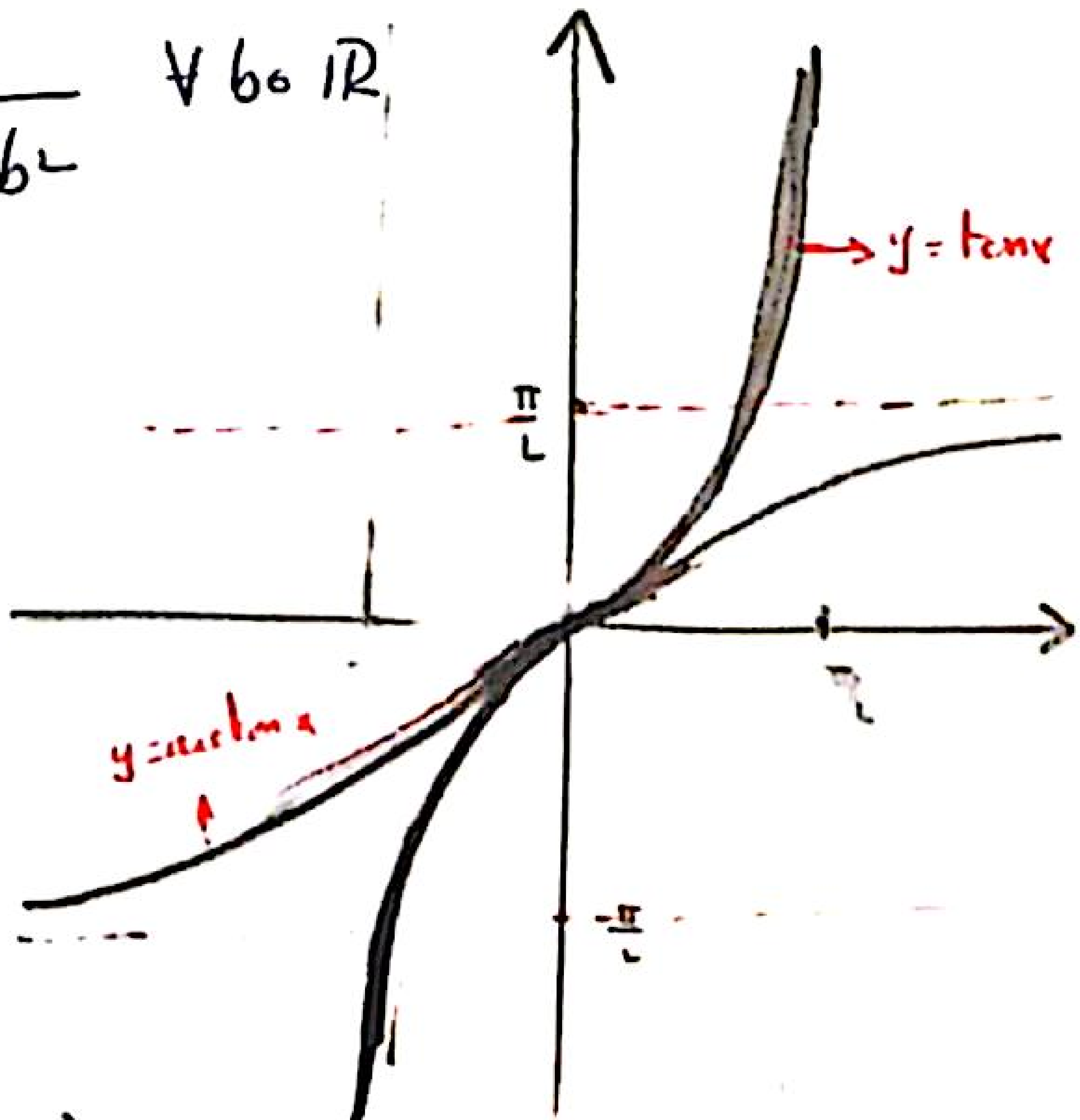
x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\to +\infty$
$\arctan x$	0	$\frac{\pi}{6}$	$\pi/4$	$\pi/3$	$\to \pi/2$

Derivative: $\arctan'(x) = \dfrac{1}{1+x^2} \quad \forall x \in \mathbb{R}$.

Let $b = f(a) = \tan a \Rightarrow f'(a) = 1 + \tan^2(a) > 0$

$$f^{-1\prime}(b) = \frac{1}{f'(a)} = \frac{1}{1+\tan^2 a} = \frac{1}{1+b^2} \quad \forall b \in \mathbb{R}$$

. $\arctan'(u(x)) = \dfrac{u'(x)}{1+u^2(x)}$



Exercise:

① Show that $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ $\forall x > 0$

② By applying M.V.T, show that:

$$\lim_{t \to 0^+} \frac{\arctan t}{t} = 1$$

③ Deduce $\lim_{x \to +\infty} (\pi x - 2x\arctan x)$.

sol ① $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ $\forall x > 0$

Let $f(x) = \arctan x + \arctan \frac{1}{x}$.

$$f'(x) = \frac{1}{1+x^2} + \frac{\left(\frac{-1}{x^2}\right)}{1+\left(\frac{1}{x}\right)^2} = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \quad \forall x > 0$$

So $f'(x) = 0$ $\forall x > 0 \Rightarrow f(x) = k$, $k \in \mathbb{R}$ $\forall x > 0$

$\Rightarrow \arctan x + \arctan\left(\frac{1}{x}\right) = k$ $\forall x > 0$.

Let $x = 1 \Rightarrow \arctan 1 + \arctan 1 = k \Rightarrow k = \pi/4 + \pi/4 = \pi/2$

$\Rightarrow \arctan x + \arctan\left(\frac{1}{x}\right) = \pi/2$ $\forall x > 0$

or $k = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \arctan x + \arctan\left(\frac{1}{x}\right) = 0 + \pi/2 = \pi/2$

or $k = \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \arctan x + \arctan\left(\frac{1}{x}\right) = \pi/2 + 0 = \pi/2$.

2) Apply the M.V.T to $g(x) = \arctan x$ on $[0,t]$.

- $g(x) = \arctan x$ is continuous on $[0,t]$
- $g(x) = \arctan x$ is differentiable on $]0,t[$.

So by M.V.T, there exists $c \in]0,t[$ /.

$$g(t) - g(0) = (t-0).g'(c)$$

$$\Rightarrow \arctan t = t . \frac{1}{1+c^2} \; ; \; 0 < c < t$$

$$\Rightarrow \frac{\arctan t}{t} = \frac{1}{1+c^2} \qquad 0 < c < t$$

$$\lim_{t \to 0^+} \frac{\arctan t}{t} = \lim_{t \to 0^+} \frac{1}{1+c^2} = \frac{1}{1+0} = 1. \qquad \underset{\downarrow \atop 0}{0} < \underset{\downarrow \atop 0}{c_t} < \underset{\downarrow \atop 0}{t}$$

③ $\arctan x + \arctan \left(\frac{1}{x}\right) = \frac{\pi}{2}$

$$\Rightarrow x \arctan x + x \arctan \left(\frac{1}{x}\right) = \frac{\pi}{2} x$$

$$\Rightarrow 2x \arctan x + 2x \arctan \left(\frac{1}{x}\right) = \pi x$$

$$\Rightarrow 2x \arctan \left(\frac{1}{x}\right) = \pi x - 2x \arctan x$$

$$\lim_{x \to +\infty} (\pi x - 2x \arctan x) = \lim_{x \to +\infty} 2x \arctan \left(\frac{1}{x}\right)$$

$$= 2 \lim_{x \to +\infty} x \arctan \left(\frac{1}{x}\right) = 2 \lim_{t \to 0^+} \frac{\arctan t}{t} = 2.1 = 2$$

$$\left(t = \frac{1}{x} \to 0^+ \Leftrightarrow x \to +\infty\right)$$

Exercise: Let $x \in]0,1[$.

① By M.V.T, show that:

$$1 < \frac{\arcsin x}{x} < \frac{1}{\sqrt{1-x^2}}$$

② Deduce that $\forall t \in]0, \pi/2[$, we have:

$$1 < \frac{t}{\sin t} < \frac{1}{\cos t}.$$

Sol:

① $$\frac{\arcsin x}{x} = \frac{\arcsin x - \arcsin 0}{x - 0}$$

Let $f(t) = \arcsin t$ on $[0, x]$.

· f is continuous on $[0, x]$

· f is differentiable on $]0, x[$

So by M.V.T, there exists at least $c \in]0, x[$ /

$$\arcsin x - \arcsin 0 = (x-0) \cdot \frac{1}{\sqrt{1-c^2}}$$

$$\Rightarrow \frac{\arcsin x}{x} = \frac{1}{\sqrt{1-c^2}} ; \quad 0 < c < x$$

$0 < c < x \iff 0 < c^2 < x^2 \iff -x^2 \leq -c^2 \leq 0$

$\iff 1 - x^2 \leq 1 - c^2 \leq 1$

$\iff \sqrt{1-x^2} \leq \sqrt{1-c^2} \leq 1 \iff 1 \leq \frac{1}{\sqrt{1-c^2}} \leq \frac{1}{\sqrt{1-x^2}}$

So $1 < \frac{\arcsin x}{x} < \frac{1}{\sqrt{1-x^2}}$.

2) From part 1 $\quad 1 < \frac{\arcsin x}{x} < \frac{1}{\sqrt{1-x^2}}$

Let $t = \arcsin x$, $x \in]0,1[\Leftrightarrow x = \sin t$, $t \in]0, \pi/2[$.

$$1 < \frac{\arcsin x}{x} < \frac{1}{\sqrt{1-x^2}} \Rightarrow 1 < \frac{t}{\sin t} < \frac{1}{\sqrt{1-\sin^2 t}} \quad t \in]0, \pi/2[$$

$$\Rightarrow 1 < \frac{t}{\sin t} < \frac{1}{\sqrt{\cos^2 t}} \quad (\cos t > 0 \quad t \in]0, \pi/2[$$

$$\Rightarrow 1 < \frac{t}{\sin t} < \frac{1}{\cos t} \quad \forall t \in]0, \pi/2[.$$

Power Function:

Let $\alpha \in \mathbb{R}$ and $x \in]0,+\infty[$. the power function denoted by x^α is defined by:

$$\begin{aligned}]0,+\infty[&\longrightarrow \mathbb{R} \\ x &\longmapsto x^\alpha = e^{\alpha \ln x} \end{aligned}$$

Derivative: $(x^\alpha)' = \alpha x^{\alpha-1}$

properties: For $x>0$ and $\alpha, \beta \in \mathbb{R}$, we have:

1) $x^{\alpha+\beta} = x^\alpha \cdot x^\beta$.

2) $x^{\alpha-\beta} = \dfrac{x^\alpha}{x^\beta}$.

3) $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$

Remark: $[f(x)]^{g(x)} = e^{g \ln f(x)}$ with $f(x)>0$

* the hyperbolic functions:

1- the sine hyperbolic function denoted by shx is defined by:

$$sh(x) = \frac{e^x - e^{-x}}{2}$$

2. the cosine hyperbolic function denoted by chx is defined by:

$$chx = \frac{e^x + e^{-x}}{2}$$

3. The tangent hyperbolic function denoted by thx is defined by

$$thx = \frac{shx}{chx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties:

1) $ch^2x - sh^2x = 1$

2) $sh(2x) = 2chx.shx$

3) $ch(2x) = ch^2x + sh^2x$

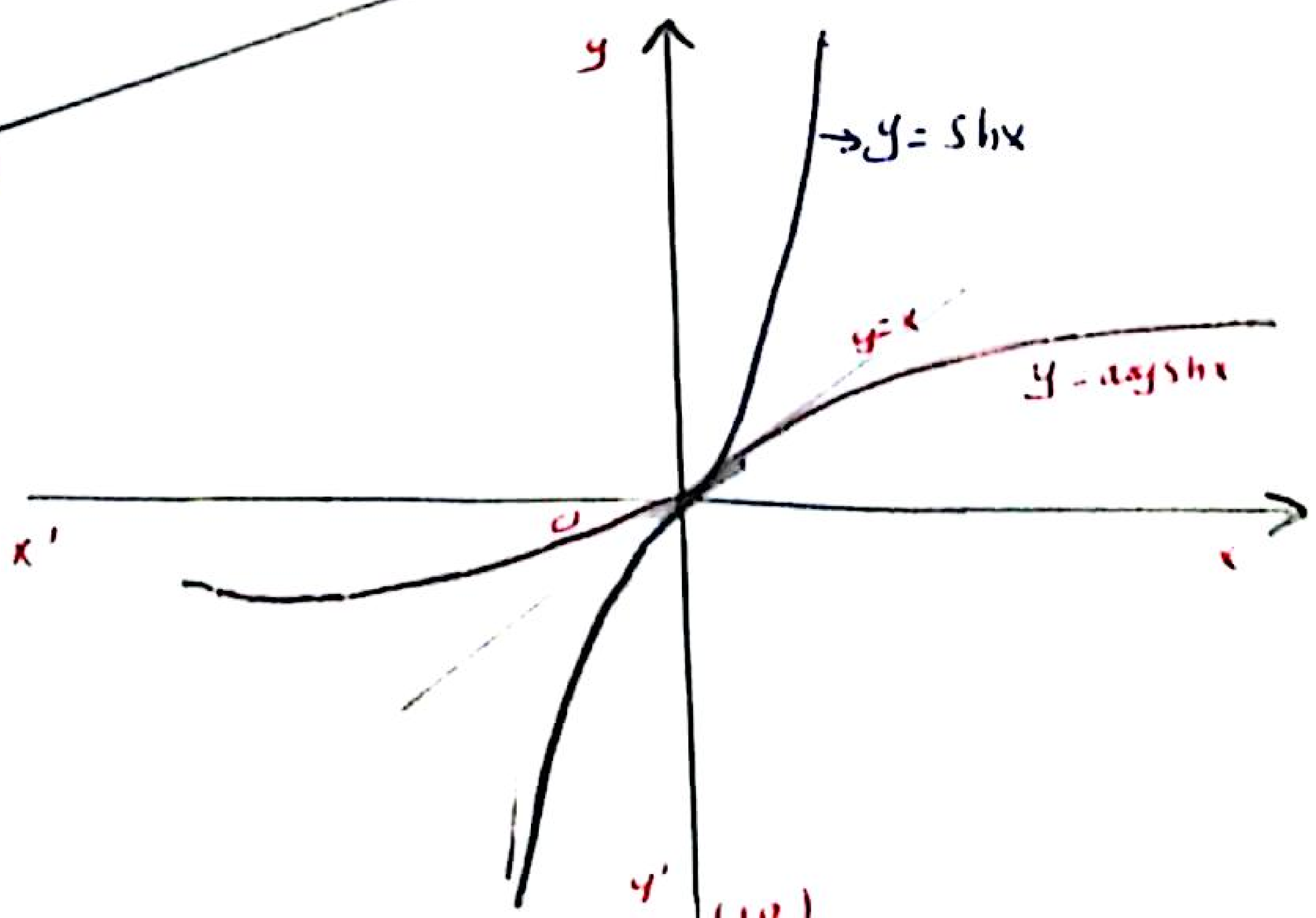
1. The sine hyperbolic function and it's inverse.

$f(x) = shx = \dfrac{e^x - e^{-x}}{2}, \ x \in \mathbb{R}.$

$f(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\left(\dfrac{e^{x} - e^{-x}}{2}\right) = -f(x)$ then $f(x) = shx$ is an odd function.

$f'(x) = sh'x = \dfrac{e^x + e^{-x}}{2} = chx > 0 \ \forall x \in \mathbb{R}. \Rightarrow \boxed{sh'x = ch\,x \ \forall x \in \mathbb{R}}$

x	0	$+\infty$
$f'(x)$	$+$	
$f(x)$	$0 \nearrow$	$+\infty$



The function $f(x) = \operatorname{sh} x$ is continuous and strictly increasing on $\mathbb{R}$ with image $\mathbb{R}$. Then it admit an inverse function called the argument sine hyperbolic function denoted by argsh s.t

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x) = \operatorname{sh} x$$

$$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow f^{-1}(x) = \operatorname{argsh} x.$$

i) $y = \operatorname{sh} x,\ x \in \mathbb{R} \Leftrightarrow x = \operatorname{argsh} y,\ y \in \mathbb{R}$

ii) $\operatorname{sh}(\operatorname{argsh} y) = y \quad \forall y \in \mathbb{R}.$ $\qquad \operatorname{sh} 0 = 0.$

iii) $\operatorname{argsh}(\operatorname{sh} x) = x \quad \forall x \in \mathbb{R}.$

Derivative: $(\operatorname{argsh} x)' = \dfrac{1}{\sqrt{1+x^2}} \quad \forall x \in \mathbb{R}.$

$$\operatorname{argsh}'(u(x)) = \frac{u'(x)}{\sqrt{1+u^2(x)}}$$

Exercise: show that $\operatorname{argsh} x = \operatorname{Ln}\left(x + \sqrt{1+x^2}\right) \quad \forall x \in \mathbb{R}.$

Sol: $f(x) = \operatorname{argsh} x \Rightarrow f'(x) = \dfrac{1}{\sqrt{1+x^2}} \quad \forall x \in \mathbb{R}.$

$$g(x) = \operatorname{Ln}\left(x + \sqrt{1+x^2}\right) \Rightarrow g'(x) = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}$$

$$\Rightarrow g'(x) = \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} \cdot \frac{1}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} \quad \forall x \in \mathbb{R}$$

so $f'(x) = g'(x) \ \forall x \in \mathbb{R} \Rightarrow f(x) = g(x) + k \quad \forall x \in \mathbb{R}.$

where k constant in $\mathbb{R}$

so $\operatorname{argsh} x = \ln(x + \sqrt{1+x^2}) + k. \quad \forall x \in \mathbb{R}.$

For $x = 0 \Rightarrow \operatorname{argsh} 0 = \ln 1 + k \Rightarrow 0 = 0 + k \Rightarrow k = 0$

$$\Rightarrow \boxed{\operatorname{argsh} x = \ln(x + \sqrt{1+x^2}) \ \forall x \in \mathbb{R}}$$

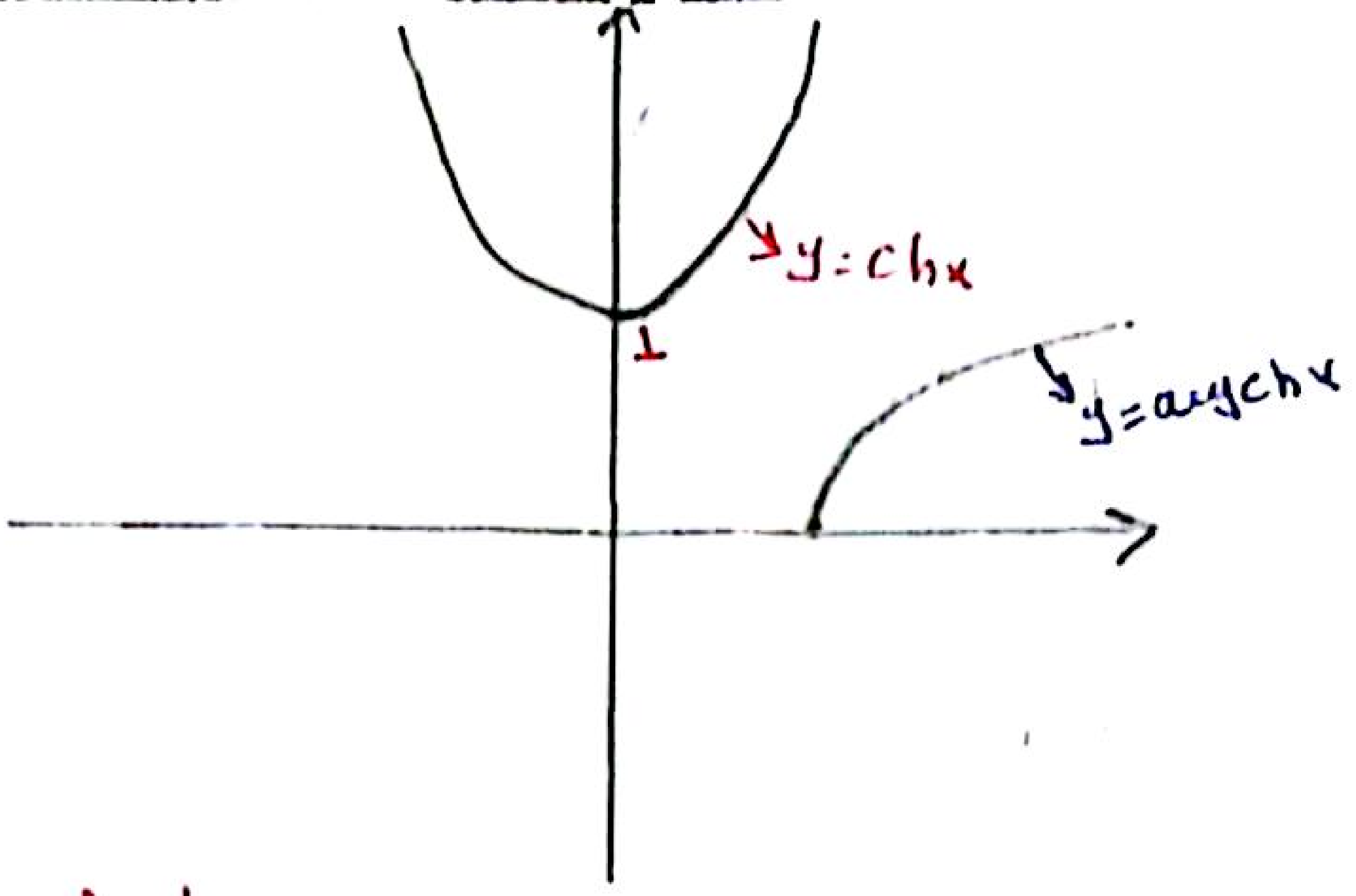
2) the cosine hyperbolic function and it's inverse

$f(x) = \operatorname{ch} x = \frac{e^x + e^{-x}}{2} \quad \forall x \in \mathbb{R}.$

$f(-x) = \frac{e^{-x} + e^x}{2} = f(x)$ so f is an even function.

$f'(x) = \frac{e^x - e^{-x}}{2} = \operatorname{sh} x \Rightarrow \boxed{\operatorname{ch}' x = \operatorname{sh} x \quad \forall x \in \mathbb{R}}$

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x) = \operatorname{ch} x$	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$



. the inverse of ch x.

the function $f(x) = ch x$ is continuous and strictly increasing on $[0, +\infty[$ with image $[1, +\infty[$ so, it admits an inverse function called the argument cosine hyperbolic function denoted by argch s.t

$$f : [0, +\infty[\longrightarrow [1, +\infty[$$
$$x \longrightarrow f(x) = ch x$$

$$f^{-1} : [1, +\infty[\longrightarrow [0, +\infty[$$
$$y \longrightarrow f^{-1}(y) = argch y .$$

i) $y = ch x,\ x \in [0, +\infty[\Leftrightarrow x = argch y,\ y \in [1, +\infty[$

ii) $ch(argch y) = y \quad \forall y \in [1, +\infty[$ $\qquad ch 0 = 1$

iii) $argch(ch x) = x \quad \forall x \in [0, +\infty[$ $\qquad argch 1 = 0$

Derivative: $\forall x \in]1, +\infty[$, $\operatorname{argch}' x = \dfrac{1}{\sqrt{x^2-1}}$

$$\operatorname{argch}'(u(x)) = \frac{u'(x)}{\sqrt{u^2(x)-1}}$$

Exercise: show that $\operatorname{argch} x = \ln(x+\sqrt{x^2-1}) \ \forall x \in [1, +\infty[$.

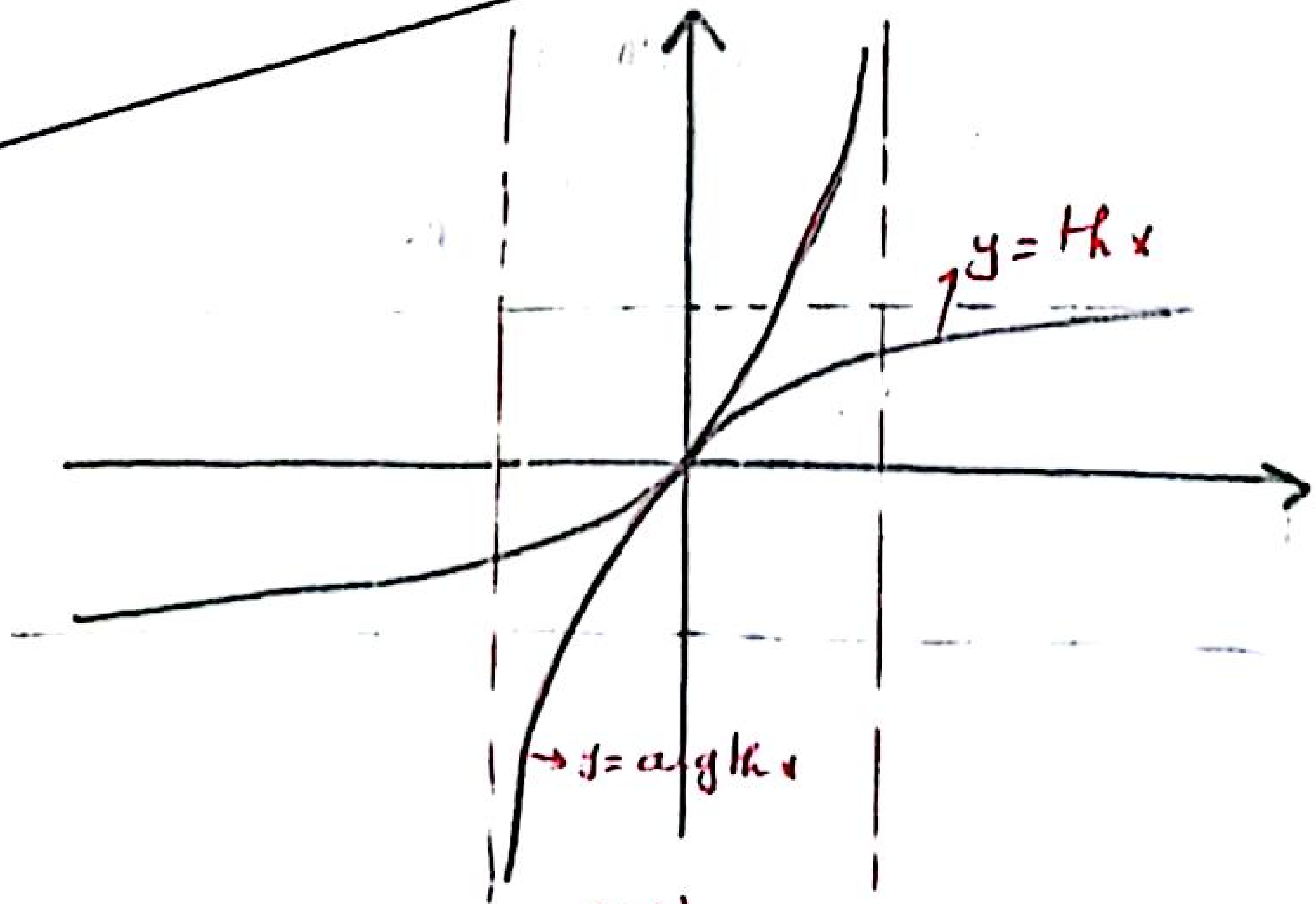
3) The tangent hyperbolic function and it's inverse.

$f(x) = \operatorname{th} x = \dfrac{\operatorname{sh} x}{\operatorname{ch} x} \quad \forall x \in \mathbb{R}.$

$f(-x) = -f(x)$ so f is an odd function

$f'(x) = \dfrac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \dfrac{1}{\operatorname{ch}^2 x} = 1 - \operatorname{th}^2 x > 0 \quad \forall x \in \mathbb{R}.$

x	$-\infty$		$+\infty$
$f'(x) = \frac{1}{\operatorname{ch}^2 x}$		$+$	
$f(x) = \operatorname{th} x$	-1	$\nearrow$	1



· Inverse of th x

The function $f(x) = \text{th}\, x$ is cont. and strictly increasing on $\mathbb{R}$ with image $]-1,+1[$ so, it admits an inverse function called the argument hyperbolic tangent denoted by argth s.t

$$\text{argth} :]-1,+1[\longrightarrow \mathbb{R}$$
$$x \longrightarrow \text{argth}\, x$$

i) $y = \text{th}\, x,\ x \in \mathbb{R} \Leftrightarrow x = \text{argth}\, y,\ y \in]-1,+1[$

ii) $\text{argth}(\text{th}\, x) = x \quad \forall x \in \mathbb{R}$ $\qquad \text{th}\, 0 = 0$

iii) $\text{th}(\text{argth}\, y) = y \quad \forall y \in]-1,+1[$ $\qquad \text{argth}\, 0 = 0$

Derivative: $\forall x \in]-1,+1[,\ \text{argth}' x = \dfrac{1}{1-x^2}$

$$\text{argth}'(u(x)) = \frac{u'(x)}{1-u^2(x)}$$

Exercise: Show that $\text{argth}\, x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad \forall x \in]-1,+1[$

Sol. $f(x) = \text{argth}\, x \Rightarrow f'(x) = \dfrac{1}{1-x^2} \quad \forall x \in]-1,+1[$

$g(x) = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right) \Rightarrow g'(x) = \dfrac{1}{2} \dfrac{u'}{u}$ where $u = \dfrac{1+x}{1-x}$

$$u'(x) = \frac{1-x+(1+x)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$\Rightarrow g'(x) = \frac{1}{2}\left[\frac{\frac{2}{(1-x)^2}}{\frac{1+x}{1-x}}\right] = \frac{1}{1-x^2} \quad \forall x \in]-1,+1[$$

So $f'(x) = g'(x)$ $\forall x \in]-1,+1[$ $\Rightarrow f(x) = g(x) + k$ $\forall x \in]-1,+1[$

$$\Rightarrow \text{argth}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + k \quad \forall x \in \mathbb{R}$$

For $x = 0 \Rightarrow$ $\text{argth}\, 0 = \frac{1}{2}\ln 1 + k \Rightarrow 0 = 0 + k \Rightarrow k = 0$

$$\Rightarrow \boxed{\text{argth}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \forall x \in]-1,+1[}$$

Exercise: Consider the function $f(x)$ defined by

$$f(x) = \arctan\left(\frac{1+x}{1-x}\right)$$

1) Determine the domain of definition of f.

2) Calculate $f'(x)$

3) What is the relation between $f(x)$ and $g(x) = \arctan x$.

Sol: $f(x) = \arctan u(x)$ where $u(x) = \dfrac{1+x}{1-x}$.

$1-x \neq 0 \iff x \neq 1$, arctan defined on $\mathbb{R}$

$\Rightarrow D_f =]-\infty, 1[\cup]1, +\infty[$.

2) $f'(x) = \dfrac{u'(x)}{1+u^2(x)}$.

$$u'(x) = \frac{1-x+1+x}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$1+u^2(x) = 1 + \left(\frac{1+x}{1-x}\right)^2 = \frac{(1-x)^2+(1+x)^2}{(1-x)^2} = \frac{2(x^2+1)}{(1-x)^2}$$

$$\Rightarrow f'(x) = \frac{u'(x)}{1+u^2(x)} = \frac{\dfrac{2}{(1-x)^2}}{\dfrac{2(x^2+1)}{(1-x)^2}} = \frac{1}{1+x^2} \quad \forall x \in]-\infty,1[\cup]1,+\infty[$$

3) $f'(x) = \dfrac{1}{1+x^2} = \arctan' x \quad \forall x \in]-\infty, 1[\cup]1, +\infty[$

let $g(x) = \arctan(x) \Rightarrow f'(x) = g'(x) \quad \forall x \in]-\infty, 1[\cup]1, +\infty[$

$$\Rightarrow f(x) = \begin{cases} g(x) + C_1 & x \in]-\infty, 1[\\ g(x) + C_2 & x \in]1, +\infty[\end{cases}$$

$$\Rightarrow \arctan\left(\frac{1+x}{1-x}\right) = \begin{cases} \arctan x + C_1 & x < 1 \\ \arctan x + C_2 & x > 1 \end{cases}$$

For $x = 0 \Rightarrow \arctan\left(\frac{1+0}{1-0}\right) = \arctan 0 + C_1$

$$\Rightarrow \pi/4 = 0 + C_1 \Rightarrow \boxed{C_1 = \pi/4}$$

For $x > 1$, $\arctan\left(\frac{1+x}{1-x}\right) = \arctan x + C_2$

$$\Rightarrow \lim_{x \to +\infty} \arctan\left(\frac{1+x}{1-x}\right) = \lim_{x \to +\infty} \arctan x + C_2$$

$$\Rightarrow \arctan -1 = \pi/2 + C_2 \Rightarrow -\pi/4 = \pi/2 + C_2$$

$$\Rightarrow \boxed{C_2 = \frac{-3\pi}{4}}$$

So

$$f(x) = \begin{cases} \arctan x + \pi/4 & \text{if } x < 1 \\ \arctan x - \frac{3\pi}{4} & \text{if } x > 1 \end{cases}$$