Derivative of a real function

Derivative at a point: Let $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function and let $a \in I$. we say that f is differentiable at a iff

$$\lim_{\substack{x \to a \\ x \neq a}} \frac{f(x) - f(a)}{x - a}$$ exists and finite.

if f is differentiable at a, we denote by:

$$f'(a) = \lim_{\substack{x \to a \\ x \neq a}} \frac{f(x) - f(a)}{x - a}.$$

Example: Let $f(x) = x^2$ and $a \in \mathbb{R}$. calculate $f'(a)$.

sol f is differentiable at a iff $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and finite.

$$\lim_{\substack{x \to a \\ x \neq a}} \frac{f(x) - f(a)}{x - a} = \lim_{\substack{x \to a \\ x \neq a}} \frac{x^2 - a^2}{x - a} = \lim_{\substack{x \to a \\ x \neq a}} \frac{(x-a)(x+a)}{(x-a)}$$

$$= \lim_{x \to a} (x+a) = 2a$$

so $f'(a) = 2a \quad \forall a \in \mathbb{R}$.

Example 2: Let $f(x) = \sqrt{x}$ calculate $f'(a)$ $\forall a > 0$.

Sol $\lim\limits_{\substack{x \to a \\ x \neq a}} \frac{f(x)-f(a)}{x-a} = \lim\limits_{\substack{x \to a \\ x \neq a}} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim\limits_{\substack{x \to a \\ x \neq a}} \frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}$

$$= \lim\limits_{\substack{x \to a \\ x \neq a}} \frac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})} = \lim\limits_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

So $f'(a) = \frac{1}{2\sqrt{a}} \quad \forall a > 0.$

Exercise: show that ① $(\sin x)' = \cos x$

② $(\cos x)' = -\sin x$.

Derivative from the Left and right

Def: We say that f is differentiable from the Left at a iff.

$\lim\limits_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ exists and finite. we denote by:

$$f'_-(a) = \lim\limits_{x \to a^-} \frac{f(x)-f(a)}{x-a}.$$

Def: we say that f is differentiable from the right at a iff

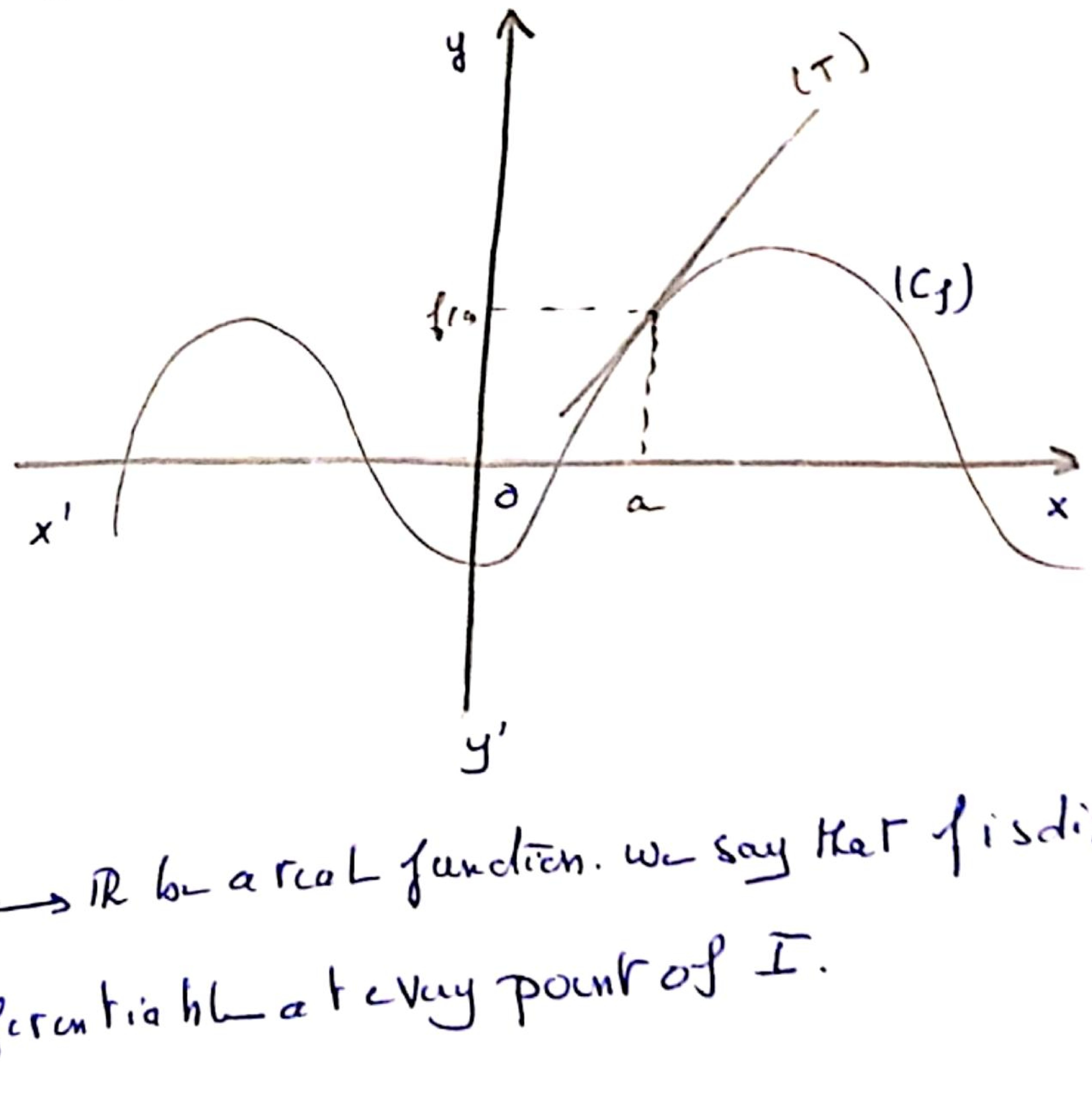
$\lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists and finite. we denote by:

$$f'_+(a) = \lim\limits_{x \to a^+} \frac{f(x)-f(a)}{x-a}.$$

Theorem: f is differentiable at a iff f is differentiable from the Left and right at a and $f'_+(a) = f'_-(a)$.

Graphically: The derivative of f at a represents the slope of the tangent to the curve (C_f) at $A(a, f(a))$. Let (T) be the tangent to (C_f) at $A(a, f(a))$, we have

$$(T) : y = f'(a)(x-a) + f(a).$$



Def: Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function. we say that f is differentiable on I iff f is differentiable at every point of I.

Exercise: Let $f(x) = |x|$.

1) Is f continuous at $x = 0$?

2) Is f differentiable at $x = 0$?

3) Conclude

Sol: $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

1) f is continuous at $x=0$ iff $\lim\limits_{x \to 0} f(x) = f(0) \Leftrightarrow \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$.

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0.$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0$

so $\lim\limits_{x \to 0} f(x) = 0$

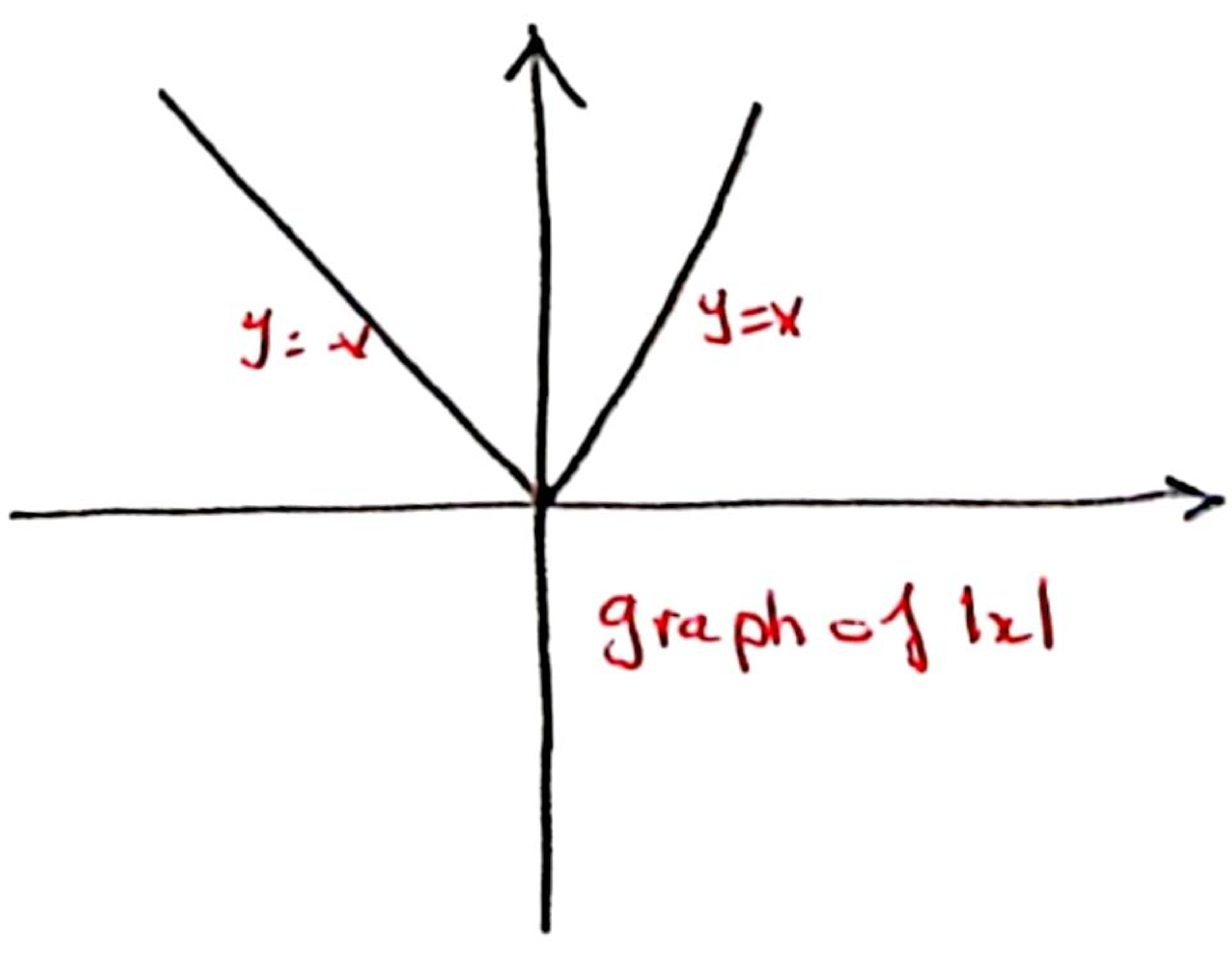
as $f(0) = 0$ then $\lim\limits_{x \to 0} f(x) = f(0)$ so f is continuous at $x=0$.

2) $\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{x - 0}{x - 0} = 1 \Rightarrow f'_+(0) = 1.$

$\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{-x - 0}{x - 0} = \lim\limits_{x \to 0^-} (-1) = -1 \Rightarrow f'_-(0) = -1.$

as $f'_+(0) \neq f'_-(0)$ so f is not differentiable at $x=0$

3) f is continuous at $x=0$ but f is not differentiable at $x=0$



Theorem: If f is differentiable at a then f is continuous at a.

but the converse is not always true.

Derivative of the usual functions:

f	f'
. cte	0
. x^n	$n x^{n-1}$
. $\sin x$	$\cos x$
. $\cos x$	$-\sin x$
. $\sqrt{x}$	$\frac{1}{2\sqrt{x}}$, $x > 0$
. e^x	e^x
. $\ln x$	$\frac{1}{x}$, $x > 0$
. $\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$

Theorem: Let u and v be two differentiable functions on I, then.

1) $(u+v)' = u' + v'$

2) $(\lambda u)' = \lambda u'$, $\lambda \in \mathbb{R}$ constant.

3) $(u \cdot v)' = u'v + v'u$

4) $\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$, $v \neq 0$

Example:

1) $f(x) = e^x + \sin x \Rightarrow f'(x) = e^x + \cos x$

2) $f(x) = \tan x = \dfrac{\sin x}{\cos x} \Rightarrow f'(x) = \dfrac{\cos x . \cos x + \sin x . \sin x}{\cos^2 x}$

$$\Rightarrow f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$\Rightarrow f'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x .$$

3) $f(x) = \sin x . e^x \Rightarrow f(x) = \cos x\, e^x + \sin x\, e^x .$

Def: A composite function is a function that depends on another function. A composite function is created when one function is substituted into another function. $u(v(x))$ is the composite function that is formed when $v(x)$ is substituted for x in $u(x)$

we denote by: $(u \circ v)(x) = u(v(x))$.

Derivative of composite functions:

$$(u \circ v)'(x) = u'(v(x)) . v'(x)$$

where u is a differentiable in $v(x)$ and v is diff. at x.

Examples:

1) $h(x) = e^{x^2}$; $f(x) = e^x$, $g(x) = x^2$.

$(f \circ g)(x) = f(g(x)) = f(x^2) = e^{x^2} \Rightarrow h(x) = (f \circ g)(x)$.

2) $h(x) = \ln(1+x^2)$; $f(x) = \ln x$, $g(x) = 1+x^2$.

$(f \circ g)(x) = f(g(x)) = f(1+x^2) = \ln(1+x^2) = h(x)$.

Example: Calculate the derivative of the following functions:

1) $f(x) = \ln(1+x^2)$; $u(x) = \ln x$, $V(x) = 1+x^2$

$f(x) = (u \circ V)(x) = u(V(x))$

$\Rightarrow f'(x) = u'(V(x)) \cdot V'(x)$.

$u'(x) = \frac{1}{x}$, $V'(x) = 2x$

$\Rightarrow f'(x) = \frac{1}{V(x)} \cdot V'(x) = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}$

Note: $(\ln u(x))' = \frac{u'(x)}{u(x)}$, $u \neq 0$, $u(x) > 0$.

. $(e^{u(x)})' = u'(x)\, e^{u(x)}$.

. $\sin(u(x))' = u'(x) \cos(u(x))$

. $\cos(u(x))' = -u'(x) \sin(u(x))$

. $\sqrt{u(x)} = \frac{u'(x)}{2\sqrt{u(x)}}$

Example2: $f(x) = \tan(e^x)$

$$f'(x) = \left(1 + \tan^2(e^x)\right) . e^x .$$

3) $f(x) = \ln(1 + \cos x)$.

$$f'(x) = \frac{u'(x)}{u(x)} \text{ where } u(x) = 1 + \cos x ,\ u'(x) = -\sin x$$

$$\Rightarrow f'(x) = \frac{-\sin x}{1 + \cos x} .$$

Exercise: Let $a \in \mathbb{R}$ and f be a real function defined by:

$$f(x) = \begin{cases} \sqrt{1+x} & \text{if } x > 0 \\ \dfrac{x+a}{x-2} & \text{if } x < 0 . \end{cases}$$

1) Determine "a" so that f is extended by continuity at $x = 0$.

Let g be the extension by continuity of f at 0.

2) Determine $g'(x)$ if $x \neq 0$.

3) Is g differentiable at $x = 0$? Justify.

Sol: f is extendable by continuity at $x=0$ iff $\lim_{x\to 0} f(x)$ exists and finite.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\sqrt{1+x}\right) = \sqrt{1+0} = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x+a)}{x-2} = \frac{-a}{2}$$

So f is extendable by continuity at $x=0$ iff $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$\Leftrightarrow \frac{-a}{2} = 1 \Rightarrow \boxed{a = -2}$

Let g be the extension of f $(a=-2)$ then g is given by:

$$g(x) = \begin{cases} \sqrt{1+x} & x > 0 \\ 1 & \text{if } x = 0 \\ 1 & \text{if } x < 0. \end{cases}$$

2) $g'(x)$? $\forall x \neq 0$

For $x > 0 \Rightarrow g(x) = \sqrt{1+x} \Rightarrow g'(x) = \frac{1}{2\sqrt{1+x}}$ $\forall x > 0$.

For $x < 0 \Rightarrow g(x) = 1 \Rightarrow g'(x) = 0$ $\forall x < 0$.

3) g is diff. at 0 ?

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+} \frac{\sqrt{1+x}-1}{x}$$

$$= \lim_{x\to 0^+} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0^+} \frac{(\sqrt{1+x})^2-1^2}{x(\sqrt{1+x}+1)} = \lim_{x\to 0^+} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0^+} \frac{x}{x(\sqrt{1+x}+1)} = \lim_{x\to 0^+} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

so $\boxed{g'_+(0) = \frac{1}{2}}$

. $$\lim_{x\to 0^-} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^-} \frac{1-1}{x} = \lim_{x\to 0^-} 0 = 0$$

so $\boxed{g'_-(0) = 0}$.

as $g'_+(0) \neq g'_-(0)$ then g is not differentiable at $x=0$.

Exercise: Let $f(x) = (x+2)e^{-x^2}$.

1) calculate $f'(x)$ $\forall x \in \mathbb{R}$. Deduce $f'(0)$?

2) Deduce $\lim_{x\to 0} \frac{(1-(x+1)e^{-x^2})}{x-0}$. Note that $f(0)=2$.

Sol.

1) $f(x) = (x+1)e^{-x^2} \Rightarrow f'(x) = e^{-x^2} + (x+1)(-2x)e^{-x^2}$

$\Rightarrow f'(x) = e^{-x^2}\left(1 - 2x(x+1)\right) = e^{-x^2}\left(1 - 2x - 2x^2\right) \quad \forall x \in \mathbb{R}$

$\Rightarrow f'(0) = e^{0}(1) = 1.$

2) $$\lim_{x \to 0} \frac{1 - (x+1)e^{-x^2}}{x} = \lim_{x \to 0} \frac{f(0) - f(x)}{x - 0}$$

$$= -\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = -f'(0) = -1.$$

Derivative of highest order:

Let f be a real function of the real variable x defined over the interval I.

If f admits a derivative on I and if it's derivative function f' is differentiable over I, then the derivative of f' denoted by f'' is called the second derivative of f over I.

we define, by mathematical induction, the derivative of order n of f denoted by $f^{(n)}$ over I by:

$$f^{(n)} = \left(f^{(n-1)}\right)' \quad \forall n \in \mathbb{N}^*$$

then $f' = f^{(1)}$, $(f')' = f'' = f^{(2)}$, $(f^{(2)})' = f^{(3)}$

.... $(f^{(n-1)})' = f^{(n)}$.

Example: $f(x) = x^3 \Rightarrow f^{(1)}(x) = 3x^2 \Rightarrow f^{(2)}(x) = 6x \Rightarrow f^{(3)}(x) = 6$

$\Rightarrow f^{(4)}(x) = 0 \Rightarrow f^{(4)}(x) = f^{(n)}(x) = 0 \quad \forall n \geq 4$.

Applications:

1. Rolle's theorem:

Let $f: [a,b] \longrightarrow \mathbb{R}$. such that:

- f is continuous on $[a,b]$.
- f is differentiable on $]a,b[$
- $f(a) = f(b)$.

then there exists at least $c \in]a,b[$ such that $f'(c) = 0$

Graphically: there exists a point $(c, f(c))$ in (C_f) such that the tangent to the curve (C_f) at $C(c, f(c))$ is parallel to x-axis. indeed; let (T) be the ~~equation~~ tangent line to (C_f) at $(c, f(c))$ then

$$(T)\ y = f'(c)(x-c) + f(c) = f(c) \quad (f'(c) = 0)$$

$\Rightarrow (T) \parallel (x'ox)$

Exercise: Let $u: [0,1] \longrightarrow \mathbb{R}$ be a continuous function on $[0,1]$ and differentiable on $]0,1[$.

1) Show that $g(x) = (x^2 - x)\, e^{u(x)}$ verify the Rolle's conditions on $[0,1]$.

2) Deduce that, there exists $c \in]0,1[$ such that:

$$u'(c) = \frac{1-c}{c^2-c}.$$

Sol:

1) • $g(x)$ is continuous on $[0,1]$ as the product and composite of two continuous functions on $[0,1]$

• g is differentiable on $]0,1[$ as product and composite of differentiable functions on $]0,1[$.

$$\left.\begin{aligned} &\cdot\, g(0) = (0-0)\, e^{u(0)} = 0 \\ &g(1) = (1-1)\, e^{u(1)} = 0 \end{aligned}\right\} \Rightarrow g(0) = g(1)$$

$\Rightarrow g$ verifies Rolle's conditions on $[0,1]$.

2) as g verifies Rolle's conditions on $[0,1]$, then there exists at least $c \in]0,1[$ s.t $g'(c) = 0$

$g(x) = (x^2 - x)e^{u(x)} \Rightarrow g'(x) = (2x-1)e^{u(x)} + u'(x)(x^2-x)e^{u(x)}$

$g'(c) = 0 \Rightarrow (2c-1)e^{u(c)} + u'(c)(c^2-c)e^{u(c)} = 0$

$\Rightarrow (2c-1) + u'(c)\cdot(c^2-c) = 0$

$\Rightarrow u'(c) = \dfrac{1-2c}{c^2-c}$, $c \in]0,1[$.

Mean Value Theorem: (MVT)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real function. If

① f is continuous on $[a,b]$

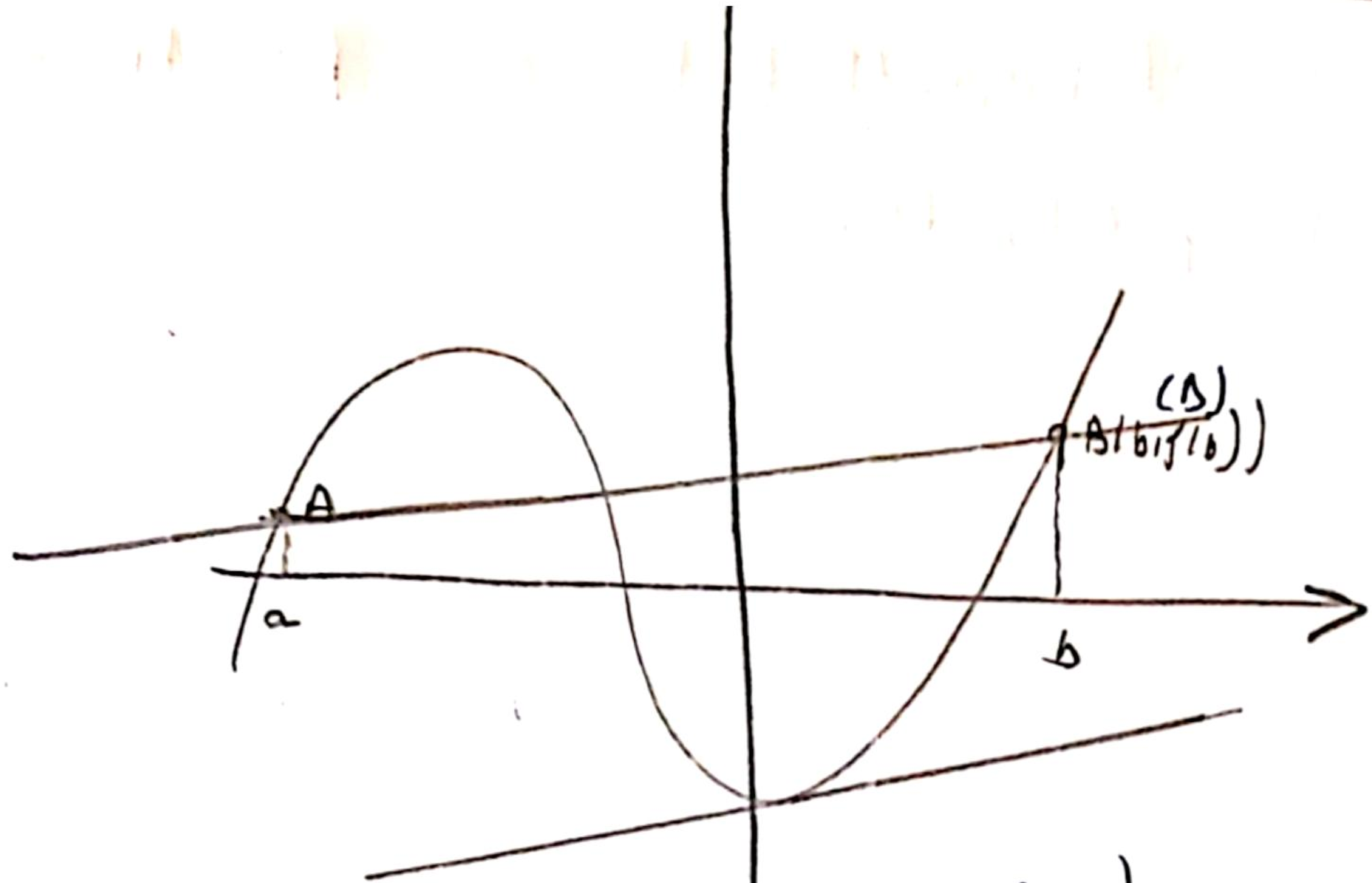
② f is differentiable on $]a,b[$

Then, there exists at least $c \in]a,b[$ such that:

$$f(b) - f(a) = f'(c)(b-a)$$

$$\Leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Graphically: Let (D) be the line passing in $A(a, f(a))$ and $B(b, f(b))$ $(D): y = \dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)$.



Let (T) be the tangent to the curve (C_f) at $C(c, f(c))$

then $(T): y = f'(c)(x-c) + f(c)$

$$y = \frac{f(b) - f(a)}{b-a}(x-c) + f(c)$$

thus, $(T) \parallel (\Delta)$.

Consequences of mean Value theorem:

1) if $f'(x) = 0$ $\forall x \in]a,b[$ then f is a constant function on $[a,b]$.

2) if $f'(x) \geq 0$ $\forall x \in]a,b[$ then f is an increasing function on $[a,b]$

3) if $f'(x) \leq 0$ $\forall x \in]a,b[$ then f is a decreasing function on $[a,b]$

4) If $f' = g' \Rightarrow f(x) = g(x) + c$ $\forall x \in [a,b]$.

Exercise: Let a and b be two real parameters. Consider the function f defined by:

$$f(x) = \begin{cases} \sqrt{1-x} & \text{if } x < 0 \\ x^2+ax+b & \text{if } 0 \leq x < 1 \\ \frac{1}{x}+\frac{1}{2} & \text{if } x \geq 1. \end{cases}$$

1) Determine a and b so that f is continuous at $x=0$ and $x=1$.

2) a) Show ~~that~~ that f is differentiable at $x=0$ and give $f'(0)$.

b) Is f differentiable at $x=1$?

c) Can we apply M.V.T to f on $[\frac{1}{2}, 3/2]$?

Sol: 1) f is continuous at $x=0$ iff $\lim_{x\to 0} f(x) = f(0)$

$$\Leftrightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0).$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{1-x} = \sqrt{1-0} = 1.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+ax+b = b.$$

$f(0) = b$

so f is cont. at $x=0 \Leftrightarrow b=1$.

* f is continuous at $x=1 \Leftrightarrow \lim\limits_{x\to 1} f(x) = f(1) \Leftrightarrow \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x) = f(1)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+ax+1) = 1+a+1 = a+2.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x}+\frac{1}{2} = 1+\frac{1}{2} = 3/2$$

$f(1) = 3/2$

so f is cont. at $x=1 \Leftrightarrow a+2 = 3/2 \Leftrightarrow a = \frac{-1}{2}$.

$$\text{so } f(x) = \begin{cases} \sqrt{1-x} & \text{if } x<0 \\ x^2-\frac{1}{2}x+1 & \text{if } 0\le x<1 \\ \frac{1}{x}+\frac{1}{2} & \text{if } x>1. \end{cases}$$

2) $$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^2-\frac{1}{2}x+1-1}{x-0}$$

$$= \lim_{x\to 0^+} \frac{x^2-\frac{1}{2}x}{x} = \lim_{x\to 0^+} \frac{\cancel{x}(x-\frac{1}{2})}{\cancel{x}}$$

$$= \frac{-1}{2}$$

$$\Rightarrow \boxed{f'_+(0) = \frac{-1}{2}}$$

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\sqrt{1-x}-1}{x}=\lim_{x\to 0^-}\frac{\sqrt{1-x}-1}{x}\cdot\frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}$$

$$=\lim_{x\to 0^-}\frac{(\sqrt{1-x})^2-1^2}{x(\sqrt{1-x}+1)}=\lim_{x\to 0^-}\frac{1-x-1}{x(\sqrt{1-x}+1)}=\lim_{x\to 0^-}\frac{-\cancel{x}}{\cancel{x}(\sqrt{1-x}+1)}$$

$$=\frac{-1}{2}\Rightarrow f'_-(0)=\frac{-1}{2}.$$

as $f'_+(0)=f'_-(0)=\frac{-1}{2}$ so f is diff. at 0 and $f'(0)=\frac{-1}{2}$.

b) • $$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^-}\frac{x^2-\frac{1}{2}x+1-\frac{3}{2}}{x-1}=\lim_{x\to 1^-}\frac{x^2-\frac{1}{2}x-\frac{1}{2}}{x-1}$$

$$=\lim_{x\to 1^-}\frac{\cancel{(x-1)}(x+\frac{1}{2})}{\cancel{(x-1)}}=\lim_{x\to 1^-}x+\frac{1}{2}=3/2,\ f'_+(1)=3/2$$

• $$\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{\frac{1}{x}+\frac{1}{2}-\frac{3}{2}}{x-1}=\lim_{x\to 1^+}\frac{\frac{1}{x}-1}{x-1}$$

$$=\lim_{x\to 1^+}\frac{1-x}{x(x-1)}=\lim_{x\to 1^+}\frac{-1}{x}=-1\Rightarrow f'_-(1)=-1$$

as $f'_+(1)\neq f'_-(1)$ so f is not diff. at $x=1$

$\Rightarrow$ f is not differentiable at $x=1\in[-\frac{1}{2},3/2]$ so we can't apply M.V.T. to f on $[-\frac{1}{2},3/2]$

Exercise: Let f be the function defined by:

$$f(x)=\begin{cases}\dfrac{3-x^2}{2} & \text{if } x<1\\[2ex] \dfrac{1}{x} & \text{if } x>1\end{cases}$$

1) Show that f is extendable by continuity at $x=1$. Let g be the extension.

2) Show that g is differentiable at $x=1$. Give $g'(1)$.

3) Show that there exists $c\in]0,2[$ such that $g(2)-g(0)=2g'(c)$

Determine all the possible values of c.

Sol: 1) f is extendable by continuity at $x=1 \Leftrightarrow \lim\limits_{x\to 1} f(x)$ exists and finite

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x}=1.$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{3-x^2}{2}=\frac{3-1}{2}=1$$

so $\lim\limits_{x\to 1} f(x)=1$. The extension g is given by:

$$g(x)=\begin{cases}\dfrac{3-x^2}{2} & \text{if } x<1\\[1ex] 1 & \text{if } x=1\\[1ex] \dfrac{1}{x} & \text{if } x>1\end{cases}$$

2) $\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{\frac{3-x^2}{2}-1}{x-1} = \lim_{x\to 1^-} \frac{3-x^2-2}{2(x-1)}$

$= \lim_{x\to 1^-} \frac{1-x^2}{2(x-1)} = \lim_{x\to 1^-} \frac{(1-x)(1+x)}{2(x-1)} = \lim_{x\to 1^-} \frac{-(1+x)}{2} = -1$

$\Rightarrow \boxed{g'_-(1) = -1}$

$\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1^+} \frac{1-x}{x(x-1)} = \lim_{x\to 1^+} \frac{-1}{x} = -1$

$\Rightarrow \boxed{g'_+(1) = -1}$

as $g'_+(1) = g'_-(1) = -1$ so g is differentiable at $x=1$ and $g'(1)=-1$

3) g is continuous on $[0,2]$

g is differentiable on $]0,2[$

so, according to MVT, there exists $c \in]0,2[$ such that

$g(2) - g(0) = (2-0)g'(c)$

$\Rightarrow g(2) - g(0) = 2g'(c) \Rightarrow \frac{1}{2} - \frac{3}{2} = 2g'(c) \Rightarrow g'(c) = \frac{-1}{2}$.

c? on $]0,1[$, $g(x) = \frac{3-x^2}{2} \Rightarrow g'(x) = -x$

$g'(c) = \frac{-1}{2} \Rightarrow -c = \frac{-1}{2} \Rightarrow \boxed{c = \frac{1}{2}}$

On $]1,2[$, $g(x)=\frac{1}{x} \Rightarrow g'(x)=\frac{-1}{x^2}$.

$g'(c)=\frac{-1}{2} \Rightarrow \frac{-1}{c^2}=\frac{-1}{2} \Rightarrow c=\sqrt{2}$ $(c>0)$.

For $x=1$, $g'(1)=-1 \neq \frac{-1}{2}$. So $c=\frac{1}{2}$ or $c=\sqrt{2}$.

Exercise: Let f be the function defined by:

$$f(x)=\begin{cases} \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x}-x & \text{if } x<0 \end{cases}$$

1) Show that f is continuous at $x=0$.

2) Study the differentiability of f at $x=0$.

3) Could MVT be applied to f on $[-1,1]$?

Exercise: By using the M.V.T, show that $\lim\limits_{t \to 0^+} \frac{\sin t}{t}=1$

Note: $f(b)-f(a)=(b-a) \cdot f'(c) \Leftrightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$

Let $t>0$, $[a,b]=[0,t]$

$f(x)=\sin x$.

. the function f is continuous on $[0,t]$

. the function f is diff. on $]0,t[$.

So by M.V.T there exists at least $c \in]0,t[$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow \frac{\sin t - \sin 0}{t-0} = \cos c_t \quad 0<c_t<t$$

$$\Rightarrow \frac{\sin t}{t} = \cos c_t \quad \text{where } 0<c_t<t$$

$$\begin{array}{ccccc} 0 & < & c_t & < & t \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

$$\lim_{t\to 0^+} \frac{\sin t}{t} = \lim_{t\to 0^+} \cos c_t = \cos 0 = 1.$$

$\Rightarrow \lim_{t\to 0^+} \frac{\sin t}{t} = 1$. Similarly if $t<0$, $\lim_{t\to 0^-} \frac{\sin t}{t} = 1$

so $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

Exercise: Consider the function g defined on $\mathbb{R}$ by:

$$g(x) = x + \ln\left(\frac{e^{-x}+1}{2}\right)$$

1) Determine $g'(x)$ $\forall x \in \mathbb{R}$.

2) Show that g' is strictly increasing.

3) Using M.V.T, Show that $\forall x>0$: $\frac{x}{2} \leq g(x) \leq \frac{xe^x}{1+e^x}$.