Limits and continuity of a real function

1. Definition and notations

Def: We call real function of one real variable x, any application f from $D \subseteq \mathbb{R}$ to $\mathbb{R}$.

$$f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x)$$

D is the domain of definition of f.

Examples ① $f(x) = \frac{1}{1-x}$

$$D_f = \{x \in \mathbb{R} \mid 1-x \neq 0\} = \{x \in \mathbb{R} / x \neq 1\} = \mathbb{R} - \{1\}$$
$$=]-\infty, 1[\cup]1, +\infty[$$

② $g(x) = \sqrt{9-x^2}$

$$D_g = \{x \in \mathbb{R} \mid 9-x^2 \geq 0\} = \{x \in \mathbb{R} \mid x^2 \leq 9\}$$
$$= \{x \in \mathbb{R} \mid -3 \leq x \leq 3\} = [-3, 3].$$

③ $h(t) = \frac{t}{\sqrt{t-1}}$

$$D_h = \{t \in \mathbb{R} / t-1 > 0\} = \{t \in \mathbb{R} / t > 1\}$$
$$=]1, +\infty[$$

Limit of a function at a point

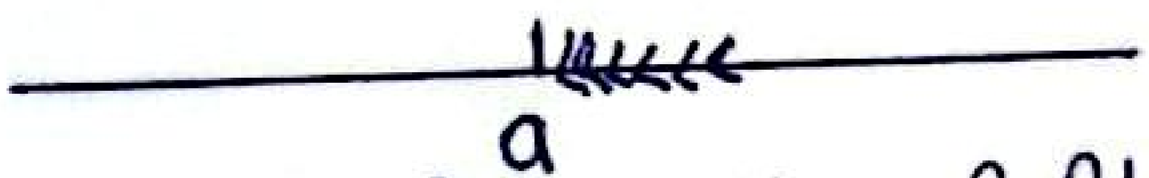
Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a real function and Let
$x \longrightarrow f(x)$ $a \in \bar{D}$ and $\ell \in \mathbb{R}$

(we suppose that a and ℓ are finite)

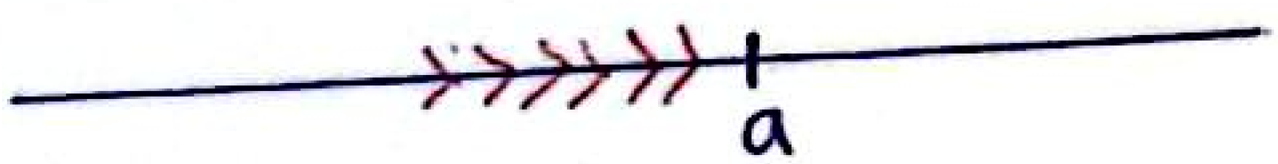
Def We say that f has a limit ℓ in $a \in \bar{D}$ and we write $\lim\limits_{x \to a} f(x) = \ell$ iff x approaches a, the values $f(x)$ tend towards ℓ.

Limit Left and right

1. We say that f has a limit ℓ on the right in a and we write $\lim\limits_{x \to a^+} f(x) = \ell$ iff when x is close to a but x greater than a then $f(x)$ tends to ℓ.



2. We say that f has a Limit ℓ on the left in a and we write $\lim\limits_{x \to a^-} f(x) = \ell$ iff when x tends to a but $x < a$ then $f(x)$ tends to ℓ.

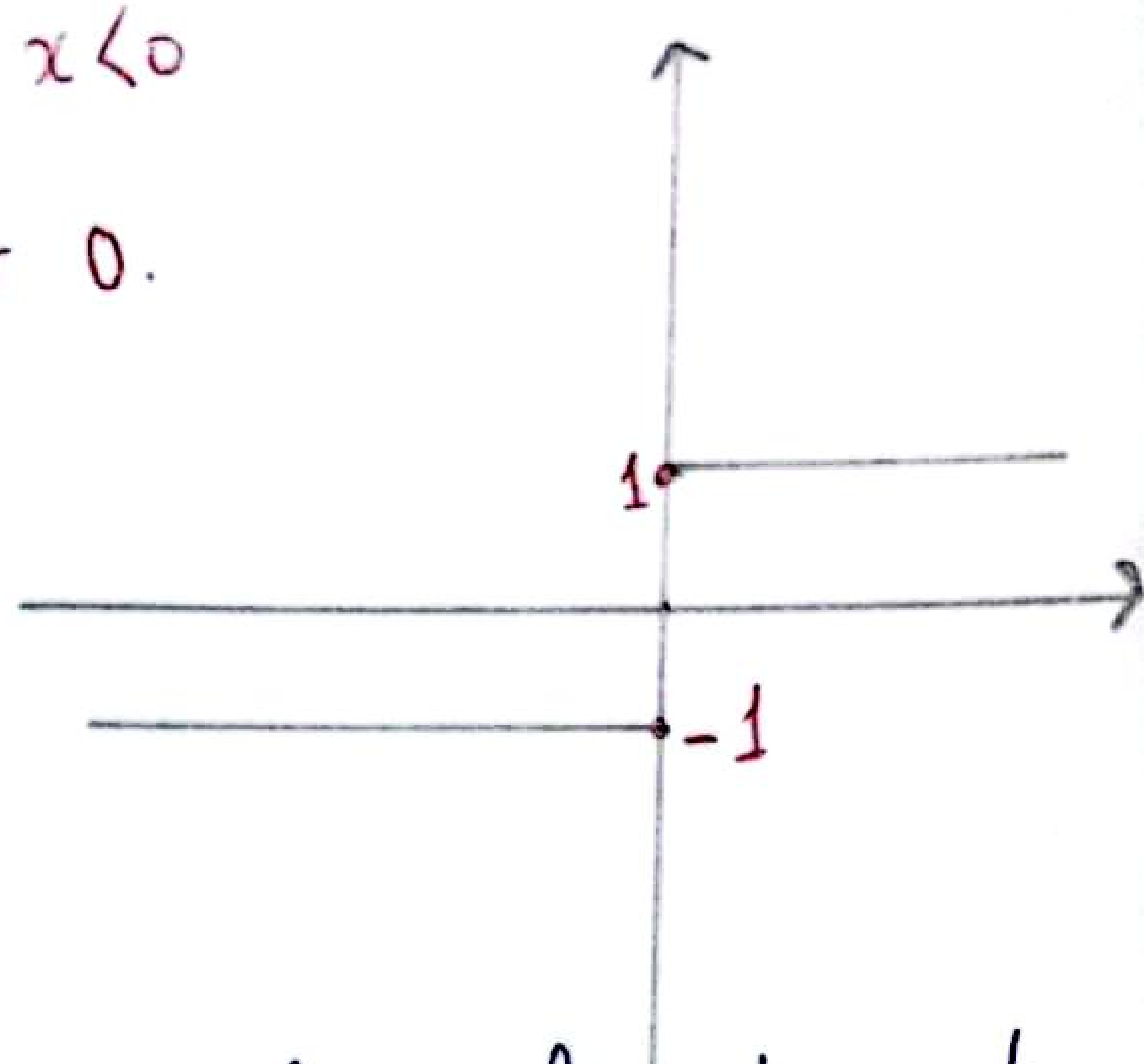


theorem $\lim\limits_{x \to a} f(x) = \ell \iff \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = \ell$.

Note: If $\lim\limits_{x \to a^+} f(x) \neq \lim\limits_{x \to a^-} f(x)$ then $\lim\limits_{x \to a} f(x)$ does not exist.

Ex Let $f(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$

study the limit of f at 0.



Sol:

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to 0^+} (1) = 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-1) = -1$

As $\lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$ then $\lim\limits_{x \to 0} f(x)$ does not exist

Properties if $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = m$

(L, a, m are finite)

then

1) $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + m$

2) $\forall \lambda \in \mathbb{R}$, $\lim\limits_{x \to a} (\lambda f(x)) = \lambda \lim\limits_{x \to a} f(x) = \lambda . L$

3) $\lim\limits_{x \to a} (f(x) . g(x)) = \lim\limits_{x \to a} f(x) . \lim\limits_{x \to a} g(x) = L . m$

4) if $m \neq 0$ $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{m}$

Note: We can generalize the notion of the limt a in the case: $a = \pm\infty$ or $L = \pm\infty$.

* We say that the function f admits as Limit L in $+\infty$ if $f(x)$ as close to L as we like by taking x sufficiently Large.

* We say that the function f tends to $+\infty$ when x tend to a if we can make $f(x)$ as large as we like by taking x sufficiently close to a and we write $\lim\limits_{x \to a} f(x) = +\infty$.

. We define in a similar way:

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

Note : the indeterminate form are :

$$0 \times \infty \ , \ \frac{\infty}{\infty} \ , \ \frac{0}{0} \ , \ \infty - \infty \ , \ 1^{\infty} \ , \ \ldots$$

Exercise : Calculate the following limits :

1. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{0}{0} \text{ ind. f.}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x \to 0} \frac{(\overset{a}{\sqrt{1+x}} - \overset{b}{\sqrt{1-x}})(\overset{a}{\sqrt{1+x}} + \overset{b}{\sqrt{1-x}})}{x(\sqrt{1+x} + \sqrt{1-x})}$$

(4)

$$= \lim_{x \to 0} \frac{(\sqrt{1+x})^2 - (\sqrt{1-x})^2}{x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{1+x-(1-x)}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = \frac{2}{\sqrt{1}+\sqrt{1}}$$

$$= \frac{2}{2} = 1.$$

2) $\lim_{x \to 0} \frac{\sqrt{1-x} - 1}{|x|}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0} \frac{\sqrt{1-x} - 1}{|x|} = \frac{0}{0} \text{ ind. f.}$$

$$\lim_{x \to 0} \frac{\sqrt{1-x} - 1}{|x|} = \lim_{x \to 0} \frac{(\overset{a}{\sqrt{1-x}} \overset{-}{-} \overset{b}{1})(\overset{a}{\sqrt{1-x}} \overset{+b}{+1})}{|x|(\sqrt{1-x} + 1)}$$

$$= \lim_{x \to 0} \frac{(\sqrt{1-x})^2 - (1)^2}{|x|(\sqrt{1-x} + 1)} = \lim_{x \to 0} \frac{1-x-1}{|x|(\sqrt{1-x} + 1)}$$

$$= \lim_{x \to 0} \frac{-x}{|x|(\sqrt{1-x} + 1)}.$$

$$\lim_{x \to 0^+} \frac{-x}{|x|(\sqrt{1-x} + 1)} = \lim_{x \to 0^+} \frac{-x}{x(\sqrt{1-x} + 1)}$$

$$= \lim_{x \to 0^+} \frac{-1}{\sqrt{1-x} + 1} = \frac{-1}{2}.$$

$$\lim_{x\to 0^+} \frac{-x}{|x|(\sqrt{1-x}+1)} = -\frac{1}{2}$$

$$\lim_{x\to 0^-} \frac{-x}{-x(\sqrt{1-x}+1)} = \lim_{x\to 0^-} \frac{1}{\sqrt{1-x}+1} = \frac{1}{2}$$

Like $\lim_{x\to 0^+} \frac{\sqrt{1-x}-1}{|x|} \neq \lim_{x\to 0^-} \frac{\sqrt{1-x}-1}{|x|}$

So $\lim_{x\to 0} \frac{\sqrt{1-x}-1}{|x|}$ does not exist.

3) $\lim_{x\to 5} \underbrace{\frac{\sqrt{x^2-10x+25}}{x-5}}_{f(x)} = \frac{0}{0}$ ind. F.

Note: $\sqrt{t^2} = |t|$

$$\lim_{x\to 5} \frac{\sqrt{x^2-10x+25}}{x-5} = \lim_{x\to 5} \frac{\sqrt{(x-5)^2}}{(x-5)} = \lim_{x\to 5} \frac{|x-5|}{x-5}$$

$$\lim_{x\to 5^+} \frac{|x-5|}{(x-5)} = \lim_{x\to 5^+} \frac{(x-5)}{(x-5)} = 1.$$

$$\lim_{x\to 5^-} \frac{|x-5|}{(x-5)} = \lim_{x\to 5^-} \frac{-(x-5)}{(x-5)} = -1.$$

Like $\lim_{x\to 5^+} f(x) \neq \lim_{x\to 5^-} f(x)$ so $\lim_{x\to 5} f(x)$ does not exist.

4) $\lim_{x\to 0} \frac{\sin x}{x^2-x} = \frac{0}{0}$ ind. F.

Note: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{\sin x}{x^2 - x} = \lim_{x\to 0} \frac{\sin x}{x(x-1)} = \lim_{x\to 0}\left(\frac{\sin x}{x} \cdot \frac{1}{x-1}\right)$$

Like $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1}{x-1} = -1$

So $\lim_{x\to 0} \frac{\sin x}{x^2-x} = 1 \times -1 = -1$.

5) $\lim_{x\to 0} \frac{\sin(5x)}{4x} = \frac{0}{0}$ ind. F.

$$\lim_{x\to 0} \frac{\sin(5x)}{4x} = \lim_{x\to 0} \frac{\sin(5x)}{5x} \times \frac{5}{4}$$

$$= \frac{5}{4} \lim_{x\to 0} \frac{\sin(5x)}{(5x)} = \frac{5}{4} \times 1 = \frac{5}{4}$$

6) $\lim_{x\to 1} \frac{\sin(x-1)}{x^2-x} = \frac{0}{0}$ ind. F.

$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2-x} = \lim_{x\to 1} \frac{\sin(x-1)}{(x-1)\cdot x} = \lim_{x\to 1}\left(\frac{\sin(x-1)}{(x-1)} \times \frac{1}{x}\right)$$

$\lim_{x\to 1} \frac{\sin(x-1)}{x-1} = ?$ Let $t = x-1$ then $t \to 0$ when $x \to 1$

$$\Rightarrow \lim_{x\to 1} \frac{\sin(x-1)}{x-1} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

$$\Rightarrow \lim_{x\to 1} \frac{\sin(x-1)}{x^2-x} = \lim_{x\to 1} \frac{\sin(x-1)}{x-1} \times \lim_{x\to 1} \frac{1}{x}$$

$$= 1 \times 1 = 1.$$

7) $\lim_{x\to 0} |x| \cdot \sin\left(\frac{1}{x}\right)$

Note: $\lim_{t\to \infty} \sin(t)$ does not exist

$\lim_{t\to \infty} \cos(t)$ does not exist

If $x \to 0$ then $\frac{1}{x} \to \infty$ so $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist

But:
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow \quad \underbrace{-|x|}_{\substack{\downarrow \\ 0}}{}_{x \to 0} \leq |x| \sin\left(\frac{1}{x}\right) \leq \underbrace{|x|}_{\substack{\downarrow \\ 0}}{}_{x \to 0}$$

So according to the squeeze theorem

$$\lim_{x \to 0} |x| \sin\left(\frac{1}{x}\right) = 0.$$

Note: $\lim_{x \to a} \underbrace{f(x)}_{\text{bounded}} \cdot \underbrace{g(x)}_{\substack{\downarrow \\ 0}} = 0$

8) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

since $\cos\left(\frac{1}{x}\right)$ is bounded :

$$\left(-1 \leq \cos t \leq 1 \quad \forall t \in \mathbb{R}\right)$$

And $\lim_{x \to 0} x^2 = 0$ so $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

9) $\lim_{t \to \infty} \frac{\sin t}{t} = \lim_{t \to +\infty} \underbrace{\sin t}_{\text{bounded}} \times \underbrace{\frac{1}{t}}_{\to 0} = 0.$

10) $\lim_{x \to 0} \frac{1}{1+e^{\frac{1}{x}}}$

$x \to 0 \Leftrightarrow \frac{1}{x} \to \infty$ more precise : $x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty$

$x \to 0^- \Rightarrow \frac{1}{x} \to -\infty$

$\lim_{x \to 0^+} \frac{1}{1+e^{\frac{1}{x}}} = \frac{1}{\infty} = 0$ Because $\frac{1}{x} \to +\infty$ when $x \to 0^+$

$\lim_{x \to 0^-} \frac{1}{1+e^{\frac{1}{x}}} = \frac{1}{1+0} = 1$ Because $\frac{1}{x} \to -\infty$ when $x \to 0^-$

then $e^{\frac{1}{x}} \to 0$ if $x \to 0^-$

Like $\lim_{x \to 0^+} \frac{1}{1+e^{\frac{1}{x}}} \neq \lim_{x \to 0^-} \frac{1}{1+e^{\frac{1}{x}}}$

Then $\lim_{x \to 0} \frac{1}{1+e^{\frac{1}{x}}}$ dos not exist.

Continuity of a function at a point

Def: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a real function and $a \in D$. We say that f is continuous in a Iff $\lim_{x \to a} f(x) = f(a)$

f is continuous in $a \Leftrightarrow \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

Note: If f is not continuous in a, we say f is discontinuous in a.

Def :

1) We say that f is continuous to the left in a iff

$$\lim_{x \to a^-} f(x) = f(a)$$

2) We say that f is continuous to the right in a iff

$$\lim_{x \to a^+} f(x) = f(a)$$

theorem : f is continuous in a iff f is continuous to the left and to the right in a.

Exercise : Let f be the function defined on $\mathbb{R}$ by :

$$f(x) = \begin{cases} \dfrac{x - |x|}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

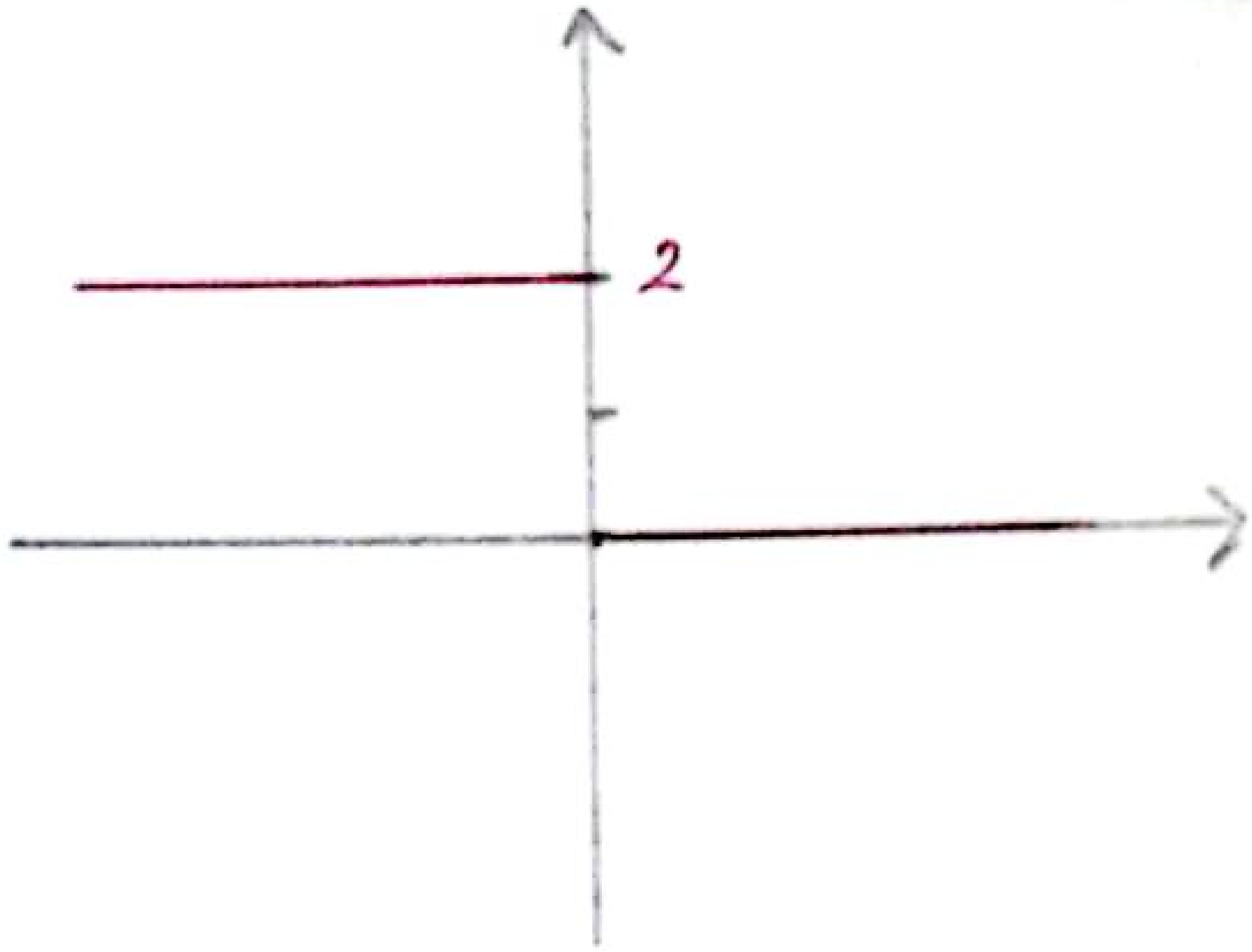
1) Find the graph of f.

2) study the Limit of f in 0

3) Is f continuous at 0 ? to the left ? to the right ?

Sol 1) $f(x) = \begin{cases} \dfrac{x - |x|}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} = \begin{cases} 0 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \end{cases}$

$$= \begin{cases} 0 & \text{if } x > 0 \\ 2 & \text{if } x = 0 \\ 2 & \text{if } x < 0 \end{cases}$$



2) $\lim_{x \to 0} f(x) = ?$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2) = 2$

Like $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$ then

$\lim_{x \to 0} f(x)$ does not exist

3) As $\lim_{x \to 0} f(x)$ does not exist then f is not continuous in 0.

$\lim_{x \to 0^-} f(x) = 2 = f(0)$ so f is continuous to the left in 0.

$\lim_{x \to 0^+} f(x) = 0 \neq f(0)$ so f is not continuous to the right in 0.

Def: Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$.
We say that f is continuous on D iff f is continuous in a $\forall a \in D$.

Theorem If f and g are two continuous functions at a then $f+g$, $f\cdot g$, λf $(\lambda \in \mathbb{R})$, $\frac{f}{g}$ $(g(x) \neq 0)$ are continuous at a.

Exercise Let f be the function defined on $\mathbb{R}^+$ by:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ ax^2 + bx + 1 & \text{if } x > 1 \end{cases}$$

where a and b are two reals numbers.

Determine a relation between a and b so that f is continuous at $x = 1$.

sol f is continuous at $x=1 \iff \lim_{x \to 1} f(x) = f(1)$

$$\iff \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2 + bx + 1) = a + b + 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (\sqrt{x}) = \sqrt{1} = 1 \quad ; \quad f(1) = \sqrt{1} = 1$$

so f is continuous at $1 \iff a + b + 1 = 1 = 1$

$$\iff a + b = 0$$

Exercise Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ and Let f be the function defined by:

$$f(x) = \begin{cases} x^2 + b & \text{if } x \leq 0 \\ \dfrac{\sin(ax)}{x} & \text{if } x > 0 \end{cases}$$

Study the continuity of f on $\mathbb{R}$ in terms of a and b.

Sol: For $x < 0$, $f(x) = x^2 + b$ continuous as a polynomial function

For $x > 0$, $f(x) = \dfrac{\sin(ax)}{x}$ is continuous as a quotient of two continuous functions

For $x = 0$: $f(x)$ is continuous at 0 $\Leftrightarrow \lim\limits_{x \to 0} f(x) = f(0)$

$$\Leftrightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + b) = b$$

$$f(0) = 0^2 + b = b$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin(ax)}{x} = \lim_{x \to 0^+} \frac{\sin(ax)}{ax} \cdot a$$

$$= a \lim_{x \to 0^+} \frac{\sin(ax)}{ax} = a \cdot 1 = a$$

So f is continuous at $0 \Leftrightarrow a=b$

Then f is continuous on $\mathbb{R}$ iff $a=b$.

Extension by continuity

Theorem: Let f be a defined and continuous functio[n] on $]a,b[\cup]b,c[$. If $\lim_{x \to b} f(x) = \ell$ $(\ell \in \mathbb{R})$ then f is extendable by continuity at point $x=b$ by the function $g(x) = \begin{cases} f(x) & \text{if } x \neq b \\ \ell & \text{if } x = b \end{cases}$

Example $f(x) = \dfrac{\sin x}{x}$

$D_f = \mathbb{R}^* =]-\infty, 0[\cup]0,+\infty[$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin x}{x} = 1$ $(\ell = 1)$

so f is extendable by continuity at $x=0$

and $g(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

the Extension of f.

Exercise Let f be a function defined by

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & \text{if } x > 0 \\ \dfrac{x}{\sqrt{4-x} - \sqrt{1-x}} & \text{if } x < 0 \end{cases}$$

Is f extendable by Continuity at point $x=0$?

Sol f is extendable by continuity at $x=0$ if $\lim_{x \to 0} f(x)$ exist and finite

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \underbrace{x^3}_{\downarrow\, 0} \underbrace{\sin\left(\frac{1}{x}\right)}_{\text{bounded}} = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{\sqrt{4-x} - \sqrt{1-x}} = \frac{0}{2-1} = 0$$

So $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$

then f is extendable by continuity at point $x=0$

and $g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ the extension of f.

Exercise Let $f(x) = \dfrac{\sqrt{1-\cos x}}{x}$

1) Determine D_f

2) Is f extendable by continuity at point $x=0$?

Sol 1) $x \neq 0$ $\quad (1-\cos x \geq 0$ because $-1 \leq \cos x \leq 1 \ \forall x \in \mathbb{R})$

So $D_f = \mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$

2) $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sqrt{1-\cos x}}{x} = \lim_{x \to 0} \frac{\sqrt{1-\cos x} \cdot \sqrt{1+\cos x}}{x(\sqrt{1+\cos x})}$$

$$= \lim_{x \to 0} \frac{\sqrt{1-\cos^2 x}}{x(\sqrt{1+\cos x})} = \lim_{x \to 0} \frac{\sqrt{\sin^2 x}}{x(\sqrt{1+\cos x})}$$

$$= \lim_{x \to 0} \frac{|\sin x|}{x(\sqrt{1+\cos x})}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|\sin x|}{x(\sqrt{1+\cos x})} = \lim_{x \to 0^+} \frac{\sin x}{x(\sqrt{1+\cos x})}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} \cdot \frac{1}{\sqrt{1+\cos x}} = 1 \cdot \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|\sin x|}{x(\sqrt{1+\cos x})} = \lim_{x \to 0^-} \frac{-\sin x}{x(\sqrt{1+\cos x})}$$

$$= -\lim_{x \to 0^-} \underbrace{\frac{\sin x}{x}}_{\downarrow \atop 1} \cdot \frac{1}{\sqrt{1+\cos x}} = -1 \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

As $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ then $\lim_{x\to 0} f(x)$ does not exist, and f is not extendable by continuity at $x=0$.

Exercise Let $f(x) = \dfrac{\sqrt{1-3x}-2}{|x+1|}$

Is f extendable by continuity at $x=-1$?

sol: $\lim_{x\to -1} f(x) = \lim_{x\to -1} \dfrac{\sqrt{1-3x}-2}{|x+1|} = \dfrac{0}{0}$ ind.F.

$= \lim_{x\to -1} \dfrac{\sqrt{1-3x}-2}{|x+1|} \cdot \dfrac{\sqrt{1-3x}+2}{\sqrt{1-3x}+2}$

$= \lim_{x\to -1} \dfrac{(1-3x)-4}{|x+1|(\sqrt{1-3x}+2)} = \lim_{x\to -1} \dfrac{-3(x+1)}{|x+1|(\sqrt{1-3x}+2)}$

$= \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \dfrac{-3(x+1)}{(x+1)(\sqrt{1-3x}+2)} = \lim_{x\to -1^+} \dfrac{-3}{\sqrt{1-3x}+2}$

$= -\dfrac{3}{4}$.

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \dfrac{-3(x+1)}{-(x+1)(\sqrt{1-3x}+2)}$

$= \dfrac{3}{4}$

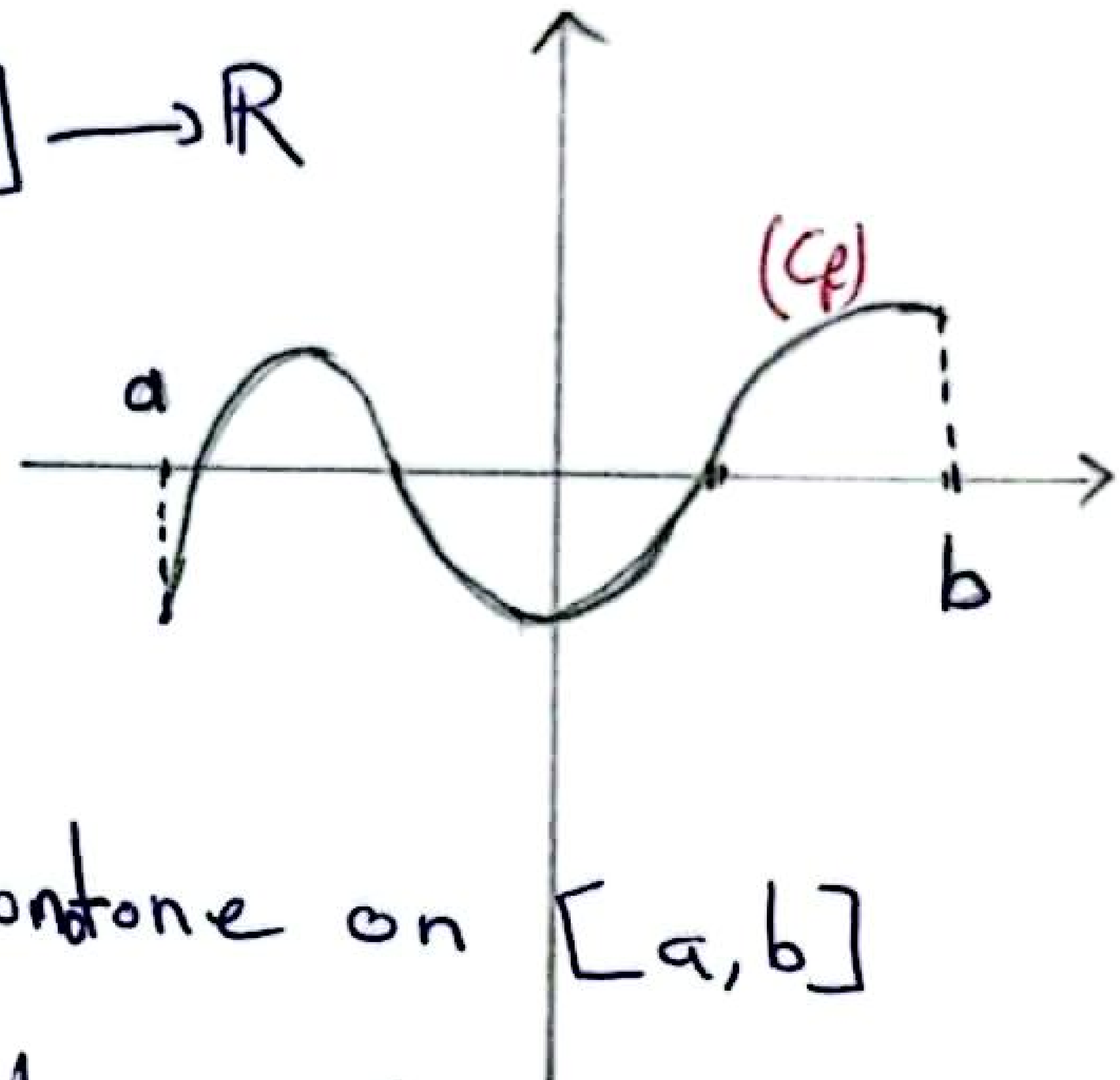
As $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$ then $\lim_{x\to -1} f(x)$ does not exist

And f is not extendable by continuity at $x = -1$

Application: Intermediate Value theorem

th: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on $[a, b]$. If $f(a) \cdot f(b) < 0$ then there is at Least $c \in]a, b[$ such that $f(c) = 0$

Corollary: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function on $[a, b]$ such that $f(a) \cdot f(b) < 0$.



If f is strictly monotone on $[a, b]$ then there is at Least a single point $\alpha \in [a, b]$ such that $f(\alpha) = 0$

Exercise: Let $f(x) = x^n + x - 1$, $n \in \mathbb{N}^*$.
show that $f(x) = 0$ admits a unique solution on $]0, 1[$.

sol f is continuous on $[0, 1]$ as a polynomial function (continuous)

$f(0) = -1 < 0$

$f(1) = 1 > 0 \Rightarrow f(0) \times f(1) < 0$

so according to the intermediate value theorem there is at Least $\alpha \in]0,1[$ Such that $f(\alpha) = 0$

but $f'(x) = nx^{n-1} + 1 > 0 \quad \forall \alpha \in]0,1[$

so f is strictly monotone (increasing) then

α is unique

Ex Let $f(x) = \dfrac{\sqrt[3]{1+\sin x} - 1}{x}$

show that f is extendable by continuity at point $x = 0$.