Finite Expansion
(near 0)

1- Definitions

→ Let $f(x)$ be a real function, $f: \mathbb{R} \to \mathbb{R}$
we say that f has a finite expansion to order n near 0 iff we can write

$$f(x) = \underbrace{P_n(x)}_{\text{polynomial of degree} \leq n} + x^n \varepsilon(x) \qquad \lim_{x \to 0} \varepsilon(x) = 0$$

→ we denote by $F.e_n(0)$ to be the finite expansion of f near 0 to order n

↳ Example
Show that near 0, the function:

$$f(x) = 1 - 2x + x^2 - x^5$$

admet a finite expansion at
1. order 1
2. order 3

Solution

1. $f(x) = P_1(x) + x\,\varepsilon(x)$

$$= \underbrace{1 - 2x}_{\text{polynomial of degre n}} + x(x - x^4) \quad \hookrightarrow \lim_{x \to 0} x - x^4 = 0$$

2. $f(x) = P_3(x) + x^3 \varepsilon(x)$

$$= 1 - 2x + x^2 + x^3(-x^2) \quad \hookrightarrow \lim_{x \to 0} -x^2 = 0$$

→ Let $a + bx + cx^2 + \dots + x^n \varepsilon(x)$, be the finite expansion of the function f near 0 to order n we have :

1. $\lim\limits_{x \to 0} f(x) = a$

then $f(x)$ is extandable by continuity at $x = 0$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

2. $g'(0) = b$

3. The equation of the tangent (T) to the representetive curve (c) of f at $x = 0$ is given by

$$(T) : Y_T = a + bx$$

4. More over the relative position of (T) and the curve (c) near $x = 0$ can be deduced from the finite expansion of f near 0 by looking at the sign of the expression $f(x) - Y_T$

$$\Rightarrow f(x) - Y_T = a_2 x^2 + a_3 x^3 + \dots + a_n x^n \varepsilon(x)$$

. if $a_2 \neq 0 \Rightarrow f(x) - Y_T \approx a_2 x^2$

↳ if $a_2 > 0$, then $f(x) - Y_T \geq 0$

$\Rightarrow$ curve (c) is above the tangente (T)

↳ if $a_2 < 0$, then $f(x) - Y_T \leq 0$

$\Rightarrow$ curve (c) is below the tangente (T)

- if $a^2 = 0$, we look at the next term in the finite expansion of $f(x)$ and we study its sign

↳ Examples:

1. Suppose $f(x) = 1 - x + 6x^2 + x^2 \varepsilon(x)$
$\Rightarrow$ (T) at $x = 0$ is given by $y_T = 1 - x$
since $F(x) - y_T \simeq 6x^2 > 0 \ \forall x$ near 0,
then the curve (C) is above (T)

2. Suppose $F(x) = 2 + x - x^3 + x^3 \varepsilon(x)$
$\Rightarrow$ (T) at $x = 0$ is given by $y_T = 2 + x$
since $F(x) - y_T \simeq -x^3$ then,
if $x > 0$ near 0, we have: $F(x) - y_T < 0 \Rightarrow C$ below T
if $x < 0$ near 0, we have: $F(x) - y_T > 0 \Rightarrow C$ above T

2- Finite Expansion of Usual functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + x^n \varepsilon(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + x^n \varepsilon(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + x^n \varepsilon(x)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + x^n \varepsilon(x)$$

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + x^n \varepsilon(x)$$

$$\sqrt{(1+x)} = (1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$\sqrt[3]{1+x} = (1+x)^{1/3} = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \frac{5}{81}x^3 + \dots$$

$$ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + x^n \varepsilon(x)$$

$$sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + x^n \varepsilon(x)$$

3- Operations on the finite expansion

Let $f(x) = P_n(x) + x^n \varepsilon_1(x)$ and $g(x) = Q_n(x) + x^n \varepsilon_2(x)$ (with $\lim_{x \to 0} \varepsilon_1(x) = 0$ and $\lim_{x \to 0} \varepsilon_2(x)$) be finite expansion to order n at 0.

→ **Sum**

$$f(x) + g(x) = P_n(x) + Q_n(x) + x^n \varepsilon(x)$$

$\varepsilon_1(x) + \varepsilon_2(x)$

$\lim_{x \to 0} \varepsilon(x) = 0$

↳ Examples

1. $f(x) = 1 - x + 2x^2 + x^2\varepsilon(x)$, $g(x) = 3 + 2x + x^2 + x^2 \varepsilon(x)$

$$f(x) + g(x) = (1 - x + 2x^2) + (3 + 2x + x^2) + x^2 \varepsilon(x)$$
$$= 4 + x + 3x^2 + x^2 \varepsilon(x)$$

2. F.e$_3$ (0) of $\ln(1+x) + \sin x$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3 \varepsilon(x)$$

$$\sin x = x - \frac{x^3}{3!} + x^3 \varepsilon(x)$$

$$\ln(1+x) + \sin x = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + \left(x - \frac{x^3}{3!}\right) + x^3 \varepsilon(x)$$

$$= 2x - \frac{x^2}{2} + \frac{x^3}{6} + x^3 \varepsilon(x)$$

→ Multiplication by a constant

$\lambda F(x) = \lambda . P_n(x) + x^n \varepsilon(x)$ $\quad \lambda \in \mathbb{R}$

↳ Example : F.e$_5$(0) of $2\cos x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^5 \varepsilon(x)$

$\Rightarrow 2.\cos x = 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + x^5 \varepsilon(x)$

$= 2 - x^2 + \frac{x^4}{12} + x^5 \varepsilon(x)$

→ Product

$\underset{\text{of order } n}{F(x) \times g(x)} = \underbrace{F(x)}_{x^p} \underbrace{g(x)}_{x^q} = \overset{\text{of order } n-q}{F(x)} \; \overset{\text{of order } n-p}{g(x)}$

where p and q are the smallest degree of x (val) in $P_n(x)$ and $Q_n(x)$ respectivly

↳ Examples :

1. F.e$_3$(0) of $\sin x . e^x$

$\sin x = x - \frac{x^3}{3!} + x^3 \varepsilon(x)$, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3 \varepsilon(x)$

$\underset{3}{\sin x . e^x} = \underbrace{\sin x}_{x^1} . \underbrace{e^x}_{x^0} = \overset{n = 3-0 = 3}{\sin x} \; e^x \rightarrow 3-1=2$

$\Rightarrow \sin x . e^x = \left(x - \frac{x^3}{6}\right)\left(1 + x + \frac{x^2}{2}\right) + x^3 \varepsilon(x)$

$= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} + x^3 \varepsilon(x)$

$= x + x^2 + \frac{x^3}{3} + x^3 \varepsilon(x)$

2. $F.e_7(0)$ of $(\cos x - 1)(\sin x - x)$

$val(\cos x - 1) = 2 \Rightarrow$ it's enough to find $F.e_5(0)$ of $\sin x - x$

$val(\sin x - x) = 3 \Rightarrow$ it's enough to find $F.e_4(0)$ of $\cos x - 1$

$$\underset{4}{\cos x - 1} = -\frac{x^2}{2!} + \frac{x^4}{4!} + x^4 \varepsilon(x)$$

$$\underset{5}{\sin x - x} = -\frac{x^3}{3!} + \frac{x^5}{5!} + x^5 \varepsilon(x)$$

$$\Rightarrow (\cos x - 1)(\sin x - x) = \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\left(-\frac{x^3}{6} + \frac{x^5}{120}\right) + x^7 \varepsilon(x)$$

$$= \frac{x^5}{12} - \frac{x^7}{240} - \frac{x^7}{144} + x^7 \varepsilon(x)$$

$$= \frac{x^5}{12} - \frac{x^7}{90} + x^7 \varepsilon(x)$$

→ **Division**

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + x^n \varepsilon(x)$

and $g(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n + x^n \varepsilon(x)$

$\Rightarrow$ the the finite expansion to order n near 0 of $\frac{f(x)}{g(x)}$ can be obtained using the euclidean division of $P_n(x)$ and $Q_n(x)$

- if there is simplification by x^p

$$\underset{n}{\frac{f(x)}{g(x)}} = \frac{f(x) \rightarrow n+p}{g(x) \rightarrow n+p}$$

→ Examples:

1. $F.e_3(0)$ of $\tan x$.

$$\underset{3}{\tan x} = \frac{\sin x}{\cos x} = \frac{x - x^3/6 + x^3\varepsilon(x)}{1 - x^2/2 + x^3\varepsilon(x)}$$

$$\begin{array}{r|l} x - \frac{x^3}{6} & 1 - \frac{x^2}{2} \\ \hline -x + \frac{x^3}{2} & x + \frac{x^3}{3} \\ \hline \frac{x^3}{3} & \end{array}$$

$$\Rightarrow \tan x = x + \frac{x^3}{3} + x^3\varepsilon(x)$$

2. $F.e_2(0)$ of $\frac{\sin x}{x}$

$n+p = 2+1 = 3$

$\frac{\sin x}{x}$ (there is devision by $x^{p=1}$)

$n=2$

$\Rightarrow \underbrace{val(x) = 1}_{p}$, so we make $F.e_3(0)$ of $\sin x$

$$\underset{3}{\sin x} = x - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$\Rightarrow \frac{\sin x}{x} = \frac{x - x^3/6 + x^3\varepsilon(x)}{x} = 1 - \frac{x^2}{6} + x^2\varepsilon(x)$$

→ **Composion**

Let $F(x) = P_n(x) + x^n\varepsilon(x)$ and $g(x) = Q_n(x) + x^n\varepsilon(x)$

$\Rightarrow$ Fog has a finite exponsion to order n near 0 of the form $P_n(Q_n(x)) + x^n\varepsilon(x)$

!! We start by finding $Q_n(x)$

↳ Examples :

1. $F.e_3(0)$ of $\ln(1+\sin x)$

$$\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x)$$

$$\ln(1+\sin x) = \ln\left(1 + x - \frac{x^3}{6} + x^3\varepsilon(x)\right) = \ln(1+t)$$

where $t = x - \frac{x^3}{6}$ $\underset{x \to 0}{t \to 0}$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + t^3\varepsilon(x)$$

$$\Rightarrow \ln(1+\sin x) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6}\right)^3 + x^3\varepsilon(x)$$

N.B : $\cdot \left(x - \frac{x^3}{6}\right)^2 = \underbrace{\left(x - \frac{x^3}{6}\right)}_{val=1}\underbrace{\left(x - \frac{x^3}{6}\right)}_{val=1} \Rightarrow n-1 = 3-1 = 2$

then it's enough to keep $F.e_2(0)$ of $\left(x - \frac{x^3}{6}\right)^2$

$\cdot \left(x - \frac{x^3}{6}\right)^3 = \underbrace{\left(x - \frac{x^3}{6}\right)}_{val=1}\underbrace{\left(x - \frac{x^3}{6}\right)^2}_{val=2} \Rightarrow n-2 = 1$

then it's enough to keep $F.e_1(0)$ of $\left(x - \frac{x^3}{6}\right)^3$

$$\Rightarrow \ln(1+\sin x) = \left(x - \frac{x^3}{6}\right) - \frac{1}{2}(x)^2 + \frac{1}{3}(x)^3 + x^3\varepsilon(x)$$

$$= x - \frac{x^3}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3\varepsilon(x)$$

$$= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\varepsilon(x)$$

2. F.e$_3$(0) of $e^{\tan x}$

$$\tan x = x + \frac{x^3}{3} + x^3 \varepsilon(x)$$

$$e^{\tan x} = e^{x + x^3/3 + x^3\varepsilon(x)} = e^t \quad \text{with } t = x + \frac{x^3}{3}$$

$$t \to 0$$
$$x \to 0$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + t^3 \varepsilon(x)$$

$$\Rightarrow e^{\tan x} = 1 + x + \frac{x^3}{3} + \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \frac{1}{6}\left(x + \frac{x^3}{3}\right)^3 + x^3\varepsilon(x)$$

$$= 1 + x + \frac{x^3}{3} + \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\varepsilon(x)$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + x^3\varepsilon(x)$$

3. F.e$_4$(0) of $\ln(1 + \cos x)$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4 \varepsilon(x)$$

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$$\ln(1 + \cos) = \ln\left(1 + 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\varepsilon(x)\right)$$

$$= \ln\left(2 - \frac{x^2}{2} + \frac{x^4}{24} + x^4\varepsilon(x)\right)$$

$$= \ln\left(2\left(1 - \frac{x^2}{4} + \frac{x^4}{48} + x^4\varepsilon(x)\right)\right)$$

$$= \ln 2 . \ln\left(1 - \frac{x^2}{4} + \frac{x^4}{48} + x^4\varepsilon(x)\right)$$

$$= \ln 2 . \ln(1 + t) \quad \text{where } t = -\frac{x^2}{4} + \frac{x^4}{48}$$

$$t \to 0$$
$$x \to 0$$

$$\ln(1+t) = t - \frac{t^2}{2} + \cancel{\frac{t^3}{3}} - \cancel{\frac{t^4}{4}} + t^4 \varepsilon(x)$$

$$\Rightarrow \ln(1+\cos x) = \ln 2 + \left(-\frac{x^2}{4} + \frac{x^4}{48}\right) - \frac{1}{2}\left(-\frac{x^2}{4} + \cancel{\frac{x^4}{48}}\right)^2 + x^4\varepsilon(x)$$

$$= \ln 2 - \frac{x^2}{4} + \frac{x^4}{48} \quad \frac{x^4}{32} + x^4 \varepsilon(x)$$

$$= \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} + x^4 \varepsilon(x)$$

!! **Remark**:

1. $\ln(a+t) = \ln\left(a(1+t/a)\right) = \ln a + \ln(1+t/a)$ with $a>0$

2. $e^{a+t} = e^a . e^t$

3. $\sqrt{a+t} = \sqrt{a(1+t/a)} = \sqrt{a} . \sqrt{1+t/a}$ with $a>0$

4. $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots$

→ **Integration**

Let f be a real function, if f has a finite expansion to order n near 0.
if F is a primitive of f, then F has a finite expansion to order $n+1$ near 0 and it's given by:

$$F(x) = F(0) + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \dots$$

↳ Example: $F.i.e_5(0)$ of $\arctan x$

$f(x) = \arctan x \quad \Rightarrow f'(x) = \frac{1}{1+x^2}$

we find the finite expansion to order 4 near 0 of $f'(x)$

$\frac{1}{1+x^2} = \frac{1}{1+t}$ where $t = x^2$ $\quad t \to 0$, $x \to 0$

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$= 1 - t + t^2 - t^3 + t^4 + t^4 \varepsilon(x)$

$\frac{1}{1+x^2} = 1 - x^2 + x^4 + x^4 \varepsilon(x)$

then by integrating the principal part we get

$\arctan x = \arctan 0 + x - \frac{x^3}{3} + \frac{x^5}{5} + x^5 \varepsilon(x)$

$= x - \frac{x^3}{3} + \frac{x^5}{5} + x^5 \varepsilon(x)$