# Primitive

## 1_ Definition and notation

. Let two functions $f$ and $F$, defined on $I$.
we say that '$F$ is a primitive of $f$' on $I$ iff:
. $F$ is differentiable on $I$
. $F'(x) = f(x) \quad \forall x \in I$

. If a function $f$ admits a primitive $F$ on $I$, The set of all primitives of $f$ on $I$ is of the form: $F + c$, where $c$ is an arbitrary constant in $\mathbb{R}$

. More simply, any primitive of the function $f$ will often be denoted: $\int f(x)\,dx$
This symbole is called 'indefinite integral of $f$'
Thus, if $F$ is a particular primitive of $f$ we have.

$$\int f(x)\,dx = F(x) + c$$

. The operation of calculating an indefinite integral is called integrating

## 2- Properties :

- $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \quad , \forall \alpha \in \mathbb{R}$
- $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

## 3- Primitives of usual functions

1) $\int a\,dx = ax + c \quad c \in \mathbb{R}$

2) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c \quad , n \neq -1$

3) $\int \overset{x^{-1}}{\dfrac{1}{x}}\,dx = \ln|x| + c \quad , \int \dfrac{1}{ax+b}\,dx = \dfrac{1}{a}\ln|ax+b| + c$

4) $\int \sqrt{x}\,dx = \frac{2}{3} x\sqrt{x} + c$

5) $\int \dfrac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$

6) $\int \dfrac{1}{x^2}\,dx = -\dfrac{1}{x} + c$

7) $\int e^{ax}\,dx = \dfrac{1}{a} e^{ax} + c \quad , \int e^x\,dx = e^x + c$

8) $\int \ln x\,dx = x\ln x - x + c$

9) $\int \sin(x)\,dx = -\cos x + c$

10) $\int \sin(ax)\,dx = \dfrac{-1}{a}\cos(ax) + c \quad , a \neq 0$

11) $\int \cos(x)\,dx = \sin x + c$

12) $\int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c \quad , a \neq 0$

13) $\int \dfrac{1}{\cos^2 x}\,dx = \tan x + c$

14) $\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$

15) $\int \frac{1}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c \quad , a \neq 0$

16) $\int \frac{1}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

17) $\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \ln\left(x + \sqrt{x^2+a}\right)$

18) $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + c$

19) $\int ch(ax) \, dx = \frac{1}{a} sh(ax) + c$

20) $\int sh(ax) \, dx = \frac{1}{a} ch(ax) + c$

21) $\int \frac{1}{ch^2 x} \, dx = th\, x + c$

## 4- Operations on Primitives

- $\int u'(x) \times u^n(x) \, dx = \frac{u^{n+1}}{n+1} + c$
- $\int \frac{u'(x)}{u(x)} \, dx = \ln |u(x)| + c$
- $\int u'(x) e^{u(x)} \, dx = e^{u(x)} + c$
- $\int \frac{u'(x)}{u^2(x)} \, dx = -\frac{1}{u} + C$

## 5- Integration Techniques

### → Change of Variables

Let $f$ be a function and we wand to find $I(x) = \int f(x)\,dx$

If $f(x)dx = g(u(x))u'(x)\,dx$ then, we can perform the change of variables $t = u(x) \Rightarrow dt = u'dx$

We write: $\int f(x)\,dx = \int g(t)\,dt$ , with $t = u(x)$

### → Integration by parts

Let $u$ and $v$ be two functions on $I$ then,

$$\int u v' dx = uv - \int u'v\,dx$$

Examples:

. $I = \int x e^x dx$

$u : x$ $\quad$ $v' : dv : e^x$

$u' : du : 1$ $\quad$ $v = \int dv = e^x$

$-\int$

$\Rightarrow I : xe^x - \int 1 . e^x dx = xe^x - e^x + c$

. $I = \int x \ln x \; dx$

$u = \ln x$ $\quad$ $v' = x$

$u' = \frac{1}{x}$ $\quad$ $v = \frac{x^2}{2}$

$$\Rightarrow I = \frac{x^2}{2}\ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2} \cdot \frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

↳ wich u we should take (أولوية ال u)

1) Inverse
2) logarithms
3) polynomials
4) Trigonometric
5) Exponential

↳ Type A:

Let $p(x)$ be a polynomial of degree $n$ , $n \in \mathbb{N}$

To calculate:

1) $\int p(x) . e^{ax} dx$

2) $\int p(x) . \sin(ax) dx$

3) $\int p(x) . \cos(ax) dx$

4) $\int p(x) . sh(ax) dx$

5) $\int p(x) . ch(ax) dx$

6) $\int p(x) . (1+ax)^{\alpha} dx$ $(\alpha \in \mathbb{R})$

we integrate $n$ times by parts and we get $u(x) = p(x)$.

!! If $n > 1$ we can use a Faster algorithm

For example for $I = \int x^2 \, sh x \, dx$ :

| derivative ($u$) | | Integral ($v'$) |
|---|---|---|
| $x^2$ | $\oplus$ | $ch x$ |
| $2x$ | $\ominus$ | $sh x$ |
| $2$ | $\oplus$ | $ch x$ |
| $0$ | | $sh x$ |

$\Rightarrow I = x^2 sh x - 2x\, ch x + 2 sh x$

↳ Type B :

Let $p(x)$ be a polynomial of degree $n$, $n \in \mathbb{N}$

To calculate:

1) $\int p(x) . \ln(ax)dx$   2) $\int p(x) . \arcsin(ax)dx$

3) $\int p(x) . \arccos(ax)dx$   4) $\int p(x) \operatorname{argsh}(ax)dx$

5) $\int p(x) . \arctan(ax)dx$   6) $\int p(x) . \operatorname{argch}(ax)dx$

7) $\int p(x) . \operatorname{argth}(ax)dx$

we integrante by parts and we get $V' = p(x)$

↳ Type C

To Calculate :

1) $\int e^{ax} . \sin(bx)dx$   2) $\int e^{ax} . \cos(bx)dx$

3) $\int e^{ax} . \operatorname{sh}(bx).dx$   3) $\int e^{ax} \operatorname{ch}(bx)dx$

we integrate by parts.

## Integrals of the form $\int \frac{P(x)}{Q(x)} dx$

$P(x)$ and $Q(x)$ are two real polynomials

1st Case: degree $P(x) <$ degree $Q(x)$

a) $I = \int \frac{1}{(x-a)^n} dx \quad a \in \mathbb{R}, n \in \mathbb{N}^*$

For $n=1 \Rightarrow I = \int \frac{1}{x-a} dx = \ln|x-a| + c$

For $n \geq 2 \Rightarrow I = \int \frac{1}{(x-a)^n} dx = \int (x-a)^{-n} dx$
$= \frac{(x-a)^{-n+1}}{-n+1} + c$

b) $I = \int \frac{\alpha x + \beta}{ax^2+bx+c} dx$

In this form we try to modify the eq to have the form $\frac{u'(x)}{u(x)}$ then we separate the rest of the eq and find it alone

Ex:

$I = \int \frac{6x+3}{x^2+2x+1} dx$

① modify to find $\frac{u'(x)}{u(x)}$

$I = 3\int \frac{2x+1}{x^2+2x+1} dx \Rightarrow 3\int \frac{(2x+1+1)-1}{x^2+2x+1} dx$

$= 3\int \frac{(2x+2)-1}{x^2+2x-1} dx = 3\underbrace{\int \frac{(2x+2)}{x^2+2x-1} dx}_{\int \frac{u'(x)}{u(x)} = \ln|u(x)|} - 3\underbrace{\int \frac{1}{x^2+2x-1} dx}_{J}$

J solve it separately

$= 3 \ln|x^2+2x-1| - 3J$

② Solve $J$

$$J = \int \frac{1}{x^2+2x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx$$

$$= \frac{(x+1)^{-2+1}}{-2+1} = \frac{(x+1)^{-1}}{-1} = -\frac{1}{(x+1)}$$

③ Add it together

$$\Rightarrow I = 3\ln|x^2+2x+1| + \frac{3}{x+1} + c$$

$2^{nd}$ Case: degree $P(x)$ > degree $Q(x)$

In this case we make an euclidean division of $p(x)$ by $Q(x)$

$$\begin{array}{l|l} P(x) & Q(x) \\ \hline & A(x) \\ \\ \hline R(x) & \end{array}$$

$$\Rightarrow P(x) = A(x).Q(x) + R(x)$$

$$\Rightarrow \frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

$$\Rightarrow I = \int \frac{P(x)}{Q(x)}\,dx = \int A(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

or

Ex

$$I = \int \frac{x^3}{x^2+1}\,dx$$

$$\begin{array}{l|l} \overset{p(x)}{x^3} & x^2+1 \;\; Q(x) \\ \hline x^3+x & x \;\; (A(x)) \\ \hline -x & \\ R(x) & \end{array}$$

$$x^3 = x(x^2+1) + (-x)$$

$$\Rightarrow \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$\Rightarrow I = \int x\,dx - \int \frac{x}{x^2+1}\,dx$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + c$$

## $3^{rd}$ Case : degree $P(x)$ = degree $Q(x)$

$$\int \frac{x^2}{x^2+5}\,dx = \int \frac{x^2+5-5}{x^2+5}\,dx = \int \frac{(x^2+5)}{(x^2+5)}\,dx - 5\int \frac{1}{x^2+5}\,dx$$

$$= x - 5\int \frac{1}{x^2+\sqrt{5}^2} = x - 5\left(\frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right)\right) + c$$

↳ **Decomposition into simple element**

we call simple elements, every element of the for $\frac{1}{(x-a)^n}$ or $\frac{\alpha x+\beta}{ax^2+bx+c}$ s.t. $D = b^2-4ac < 0$

How to decomposite $\frac{P(x)}{Q(x)}$ into Sum of simple elements

Factor denominator as completly as possible and find the partial decomposition of the rational expression. Integrate the partial fonction decomposition (P.F.D) for each factor in the denominator

| Factor of $Q(x)$ | Term in P.F.D |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $ax^2+bx+c$ | $\frac{Ax+b}{ax^2+bx+c}$ |
| $(ax+b)^K$ | $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_K}{(ax+b)^K}$ |
| $(ax^2+bx+c)^K$ | $\frac{A_1x+B_1}{ax^2+bx+c} + \dots + \frac{A_Kx+B_K}{(ax^2+bx+c)^K}$ |

Example:

$$I = \int \frac{7x^2+13x}{(x-1)(x^2+4)}\,dx$$

$$\Rightarrow \frac{7x^2+13x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x}{(x-1)(x^2+4)}$$

Set numerator equal and collect like terms

$$\Rightarrow 7x^2+13x = (A+B)x^2 + (C-B)x + 4A - C$$

Set coefficients equal to get a system and solve to get constant

$A + B = 7 \qquad C - B = 13 \qquad 4A - C = 0$

$A = 4 \qquad B = 3 \qquad C = 16$

Then, $$I = \int \frac{4}{x-1}\,dx + \int \frac{3x+16}{x^2+4}\,dx$$

$$= 4\int \frac{1}{x-1}\,dx + \frac{3}{2}\int \frac{2x}{x^2+4}\,dx + 16\underbrace{\int \frac{1}{x^2+4}\,dx}_{J}$$

$$= 4\ln|x-1| + \frac{3}{2}\ln|x^2+4 + 16J + c$$

$$J = \int \frac{1}{x^2+2^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\Rightarrow I = 4\ln|x-1| + \frac{3}{2}\ln|x^2+4| + 8\tan^{-1}\left(\frac{x}{2}\right) +$$

## Primitive of rational functions in $\sin x$ and $\cos x$

1) $\int \sin^n(x)\cos^m(x)$ : For this form we have:

a.) n odd : strip 1 sine out and convert the rest to cosines using $\sin^2(x) = 1-\cos^2(x)$ then use the substitution $t = \cos(x)$

ex: $I = \int \sin^3 x \cdot \cos^2 x \, dx$

$= \int \sin^2 x \cdot \sin x \cdot \cos^2 x \, dx$

$= \int (1-\cos^2(x)) \cdot \sin x \cdot \cos^2 x \, dx$

Let $t = \cos x$ , $dx = \dfrac{dt}{-\sin x}$

$\Rightarrow I = \int (1-t^2) \cdot t^2 \cdot \sin x \dfrac{dt}{-\sin x}$

$= \int -(1-t^2)t^2 \, dt = \int -t^2 + t^4 \, dt$

$= -\int t^2 dt + \int t^4 dt = -\dfrac{t^3}{3} + \dfrac{t^5}{5}$

$\Rightarrow -\dfrac{\cos^3 x}{3} + \dfrac{\cos^5 x}{5} + c$

b) m odd: strip 1 cosine out and convert rest to sines using $\cos^2(x) = 1 - \sin^2(x)$, the use the substitution $t = \sin(x)$

c. n and m both odd: use either a or b.

d. n and m both even: Use formulas to reduce the integral into a form that can be integrated

=> Trig-Formulas: $\sin(2x) = 2\sin(x)\cos(x)$

$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

ex:

- $I = \int \cos^2 x \, dx$

$I = \int \frac{1+\cos(2x)}{2} dx = \int \frac{1}{2} dx + \frac{1}{2}\int \cos 2x \, dx$

$= \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

- $I = \int \sin^2 x \cdot \cos^2 x \, dx$

=> $\sin(2x) = 2\sin(x)\cos(x)$

$\sin(x)\cdot\cos(x) = \frac{\sin 2x}{2}$

$\sin^2(x)\cdot\cos^2(x) = \frac{1}{4}\sin^2(2x) = \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$

$= \frac{1}{8} - \frac{1}{8}\cos(4x)$

$\Rightarrow I = \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\, dx$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x)$

2) $I = \int \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)}$ For this form we have:

$$A(x) = \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)} dx$$

a. If $A(-x) = A(x) \Rightarrow$ Change of variable $t = \cos x$
b. If $A(\pi - x) = A(x) \Rightarrow$ Change of variable $t = \sin x$
c. If $A(\pi + x) = Ax \Rightarrow$ change of variable $t = \tan x$
d. If I didn't undergoe any of these then
$t = \tan\left(\frac{x}{2}\right)$ with: $\sin = \frac{2t}{1+t^2}$, $\cos = \frac{1-t^2}{1+t^2}$

$$\tan = \frac{2t}{1-t^2}$$

$$\Rightarrow dx = \frac{2dt}{1+t^2}$$

Remark:

| | | |
|---|---|---|
| $\cos(-x) = \cos x$ | $\sin(-x) = -\sin x$ | $d(-x) = -dx$ |
| $\cos(\pi - x) = -\cos x$ | $\sin(\pi - x) = \sin x$ | $d(\pi - x) = -dx$ |
| $\cos(\pi + x) = -\cos x$ | $\sin(\pi + x) = -\sin x$ | $d(\pi + x) = dx$ |

↳ Integral of the form $I = \int \frac{f(e^x)}{g(e^x)} dx$

In this case, we make change of variables $t = e^x$
$dt = t dx$

Ex:

$$I = \int \frac{dx}{e^x + 1}, \quad t = e^x \Rightarrow dx = \frac{dt}{t}$$

$$\Rightarrow I = \int \frac{1}{t+1} \cdot \frac{dt}{t} = \int \frac{1}{t(t+1)} dt = \int \frac{(1+t) - t}{t(t+1)} dt$$

$$= \int \frac{1+t}{t(t+1)} dt - \int \frac{t}{t(t+1)} dt = \int \frac{1}{t} dt - \int \frac{1}{t+1} dt$$

$$= \ln|t| - \ln|t+1| + c \Rightarrow I = \ln|e^x| - \ln|e^x + 1| + c_1$$

$$= x - \ln(1 + e^x) + c_1$$

↳ Integrals of the form $I = \int f\left(x, \sqrt{\frac{ax+b}{cx+d}}\right) dx$

① make the change of variables $t = \sqrt{\frac{ax+b}{cx+d}}$

② Square $t$ $(t^2)$ to find $x$ in term of $t$

③ find the derivative of $x \Rightarrow dx = x' dt$

④ Replace everything and solve the integral

Ex: $I = \int \frac{1}{x} \sqrt{\frac{1-x}{1+x}}\, dx$

① Let $t = \sqrt{\frac{1-x}{1+x}}$

② $t^2 = \frac{1-x}{1+x}$

$1 - x = t^2(1+x) \Rightarrow 1 - x = t^2 + t^2 x$

$\Rightarrow 1 - t^2 = x + t^2 x \Rightarrow 1 - t^2 = x(1+t^2)$

$\Rightarrow x = \frac{1-t^2}{1+t^2}$

③ $dx = \left(\frac{1-t^2}{1+t^2}\right)' dt = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} dt$

$= \frac{-4t}{(1+t^2)^2} dt$

④ $\Rightarrow I = \int \frac{1+t^2}{1-t^2} \cdot t \cdot \frac{-4t}{(1+t^2)^2}$

- Integrals of the form $I = \int F(x, \sqrt{ax^2+bx+c})\,dx$

(A) In this case the term $ax^2+bx+c$ can be transformed into one of the following forms:

1) $\alpha(K^2-u^2)$ $(\alpha>0)$ : we make the change of variable $u = K\sin t$

2) $\alpha(K^2+u^2)$ $(\alpha>0)$ : we make the change of variable $u = K\,\mathrm{sh}\,t$ or $u = K\tan t$

3) $\alpha(u^2-K^2)$ $(\alpha>0)$ : we make the change of variable $u = K\,\mathrm{ch}\,t$ or $u = \frac{K}{\cos t}$

(B) particular case : $I = \int \frac{dx}{\sqrt{ax^2+bx+c}}$

- we transform $ax^2+bx+c$ to the form $t^2+K$ and we use the formula: $\int \frac{dt}{\sqrt{t^2+K}} = \ln\left|t+\sqrt{t^2+K}\right| + c$

or

- we transform $ax^2+bx+c$ to the form $K^2-t^2$ and we use the formula: $\int \frac{dt}{\sqrt{K^2-t^2}} = \arcsin\left(\frac{t}{K}\right) + c$

Ⓐ Ex: $I=\int\sqrt{-x^2+2x+8}\,dx$

$$-x^2+2x+8 = -[x^2-2x-8] = -[(x-1)^2-9]$$
$$= +[\underbrace{3^2}_{K^2}-\underbrace{(x-1)^2}_{u^2}]$$

$$\Rightarrow I=\int\sqrt{3^2-(x-1)^2}\,dx$$

Let $\underbrace{x-1}_{u}=\underbrace{3\sin t}_{=K\sin t} \Rightarrow x=3\sin t+1 \Rightarrow dx=3\cos t\,dt$

$$\Rightarrow I=\int\sqrt{3^2-3^2\sin^2 t}\cdot 3\cos t\,dt$$
$$=9\int\sqrt{1-\sin^2 t}\cdot\cos t\,dt$$
$$=9\int\sqrt{\cos^2 t}\cdot\cos t\,dt$$
$$=9\int\cos^2 t\,dt = 9\int\underbrace{\frac{1}{2}(1+\cos(2t))}_{\text{formula}}\,dt$$
$$=\frac{9}{2}\int 1+\cos(2t)\,dt = \frac{9}{2}t+\frac{9}{4}\sin 2t+c$$
$$=\frac{9}{2}t+\frac{9}{2}\cos t\sin t+c$$
$$=\frac{9}{2}t+\frac{9}{2}\sqrt{1-\sin^2 t}\cdot\sin t+c$$

$$x-1=3\sin t \Rightarrow \sin t=\frac{x-1}{3} \Rightarrow t=\arcsin\left(\frac{x-1}{3}\right)$$

$$\Rightarrow I=\frac{9}{2}\arcsin\left(\frac{x-1}{3}\right)+\frac{9}{2}\sqrt{1-\left(\frac{x-1}{3}\right)^2}\cdot\frac{x-1}{3}+c$$

Ⓑ Ex: $I=\int\frac{dx}{\sqrt{2x^2+4x+6}}$

$$2x^2+4x+6=2[x^2+2x+3]=2[\underbrace{(x+1)^2}_{t^2}+\underbrace{2}_{+K}]$$

$$\Rightarrow I=\int\frac{dx}{\sqrt{2}\sqrt{(x+1)^2+2}}=\frac{1}{\sqrt{2}}\underbrace{\int\frac{dx}{\sqrt{(x+1)^2+2}}}_{\int\frac{dt}{\sqrt{t^2+K}}}=\frac{1}{\sqrt{2}}\ln\left|(x+1)+\sqrt{(x+1)^2+2}\right.$$

© particular case : $I = \int \frac{\alpha x + \beta}{\sqrt{ax^2+bx+c}}\, dx$

In this case we try to find the form $\frac{(ax^2+bx+c)'}{\sqrt{ax^2+bx+c}}$

and then we have a change variable $t = ax^2+bx+c$
to obtain the form $\int \frac{dt}{\sqrt{t}}$

© Ex: $I = \int \frac{(2x+1)\,dx}{\sqrt{-2x^2+4x+8}}$

$$I = \frac{-1}{2} \underbrace{\int \frac{(-4x+4)\,dx}{\sqrt{-2x^2+4x+8}}} + 3 \underbrace{\int \frac{dx}{\sqrt{-2x^2+4x+8}}}$$

$$\int \frac{(-4x+4)\,dx}{\sqrt{-2x^2+4x+8}} = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{-2x^2+4x+8} + C$$

$$\int \frac{dx}{\sqrt{-2x^2+4x+8}} = \frac{dx}{\sqrt{-2(x^2-2x-4)}} = \int \frac{dx}{\sqrt{-2[(x-1)^2-5]}}$$

$$= \int \frac{dx}{\sqrt{2(5-(x-1)^2)}} = \frac{1}{\sqrt{2}} \underbrace{\int \frac{dx}{\sqrt{(\sqrt{5})^2-(x-1)^2}}}_{\int \frac{dt}{\sqrt{k^2-t^2}}} = \frac{1}{\sqrt{2}} \cdot \arcsin\left(\frac{x-1}{\sqrt{5}}\right) + C$$

$$\Rightarrow I(x) = -\sqrt{-2x^2+4x+8} + \frac{3}{\sqrt{2}} \arcsin\left(\frac{x-1}{\sqrt{5}}\right) + C$$

↳ Integrals of the form $\int F(x, (a^2+x^2)^\alpha)\,dx$

then we apply change of variable
$x = a\tan t$ $\quad dx = a(1+\tan^2 x)$

↳ Integrals of the form $f(x, (a^2-x^2)^\alpha)\,dx$

then we apply change of variable
$x = a\sin t$ $\quad dx = a\cos t$

6 - Difinite Integral

$$I = \int_a^b F(x)\,dx = [\underbrace{G(x)}_{\text{primitive of } F(x)}]_a^b = G(b) - G(a)$$

- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$