Limits and Continuity of a real Function

1_ Limit of a function

→ Definition

. $\lim_{x \to a} f(x) = \ell$ (at a point)

. $\lim_{x \to a^+} f(x) = \ell$ (on the right)

. $\lim_{x \to a^-} f(x) = \ell$ (on the left)

→ Existence of a Limit

The function f admits a limit at 'a' iff:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \ell \implies \lim_{x \to a} f(x) = \ell$$

→ Properties:

1) $\lim_{x \to a} \left(f(x) \pm g(x)\right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

2) $\lim_{x \to a} \left(f(x) . g(x)\right) = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

3) $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$ $\left(\lim_{x \to a} g(x) \neq 0\right)$

4) $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$ 5) $\lim_{x \to a} (\underbrace{c}_{\text{constant}} f(x)) = c \lim_{x \to a} f(x)$

Basic Limit Evaluations

1) $\lim\limits_{x \to \pm\infty} \dfrac{a}{x^n} = 0$

2) $\lim\limits_{x \to \infty} x^n = \infty$

3) $\lim\limits_{x \to 0^+} \dfrac{a}{x} = \infty$

3) $\lim\limits_{x \to 0^-} \dfrac{a}{x} = -\infty$

Evaluation Techniques

1) Factor and Cancel

$$\lim_{x \to 2} \frac{x^2+4x-12}{x^2-2x} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+6)}{x\cancel{(x-2)}} = \lim_{x \to 2} \frac{x+6}{x} = 4$$

2) Conjugate (Rationalize Numerator/Denominator)

$$\lim_{x \to 9} \frac{3-\sqrt{x}}{x^2-81} = \lim_{x \to 9} \frac{3-\sqrt{x}}{x^2-81} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x \to 9} \frac{9-x}{(x^2-81)(3+\sqrt{x})}$$

$$= \lim_{x \to 9} \frac{-\cancel{(x-9)}}{\cancel{(x-9)}(x+9)(3+\sqrt{x})} = \lim_{x \to 9} \frac{-1}{(x+9)(3+\sqrt{x})} = \frac{-1}{(18)(6)} = \frac{-1}{108}$$

3) Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim\limits_{x \to \pm\infty} p(x)/q(x)$, factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x \to -\infty} \frac{3x^2-4}{5x-2x^2} = \lim_{x \to -\infty} \frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)} = \lim_{x \to -\infty} \frac{3-\cancel{\frac{4}{x^2}}_{0}}{\cancel{\frac{5}{x}}_{0}-2} = -\frac{3}{2}$$

4) Combine Rational Expressions

$$\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h \to 0} \frac{1}{h} \frac{x-(x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

5) Piecewise Function

$\lim_{x \to -2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \geq -2 \end{cases}$

Compute two one sided limits

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x^2+5 = 9$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} 1-3x = 7$$

$\lim_{x \to -2^-} g(x) \neq \lim_{x \to -2^+} g(x)$. So $\lim_{x \to -2} g(x)$ does not exist

If the two one sided limits had been equal then $\lim_{x \to -2} g(x)$ would have existed and had the same value

6) L'Hospital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

(a is a number, ∞ or $-\infty$)

→ Indeterminate Forms:

$0 \times \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, 1^{∞}...

Notes:

- $\sqrt{t^2} = |t|$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to \infty} \sin(t)$ and $\lim_{x \to \infty} \cos(t)$ Does not Exist
- $\lim \underbrace{F(x)}_{\text{bounded}} \cdot \underbrace{g(x)}_{\text{tends toward } 0} = 0$

$\lim_{x \to 0} x^2 \cos(1/x)$: sin $\cos(1/x)$ is bounded
($-1 \leq \cos t \leq 1 \; \forall t \in \mathbb{R}$)
And $\lim_{x \to 0} x^2 = 0$

$\Rightarrow \lim_{x \to 0} x^2 \cos(1/x) = 0$

- $\sin(x+h) = \sin x \cos h + \cos x \sin h$
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
- $\cos(x+h) = \cos x \cos h - \sin x \sin h$
- $ax^2 + bx + c = a(x - x')(x - x'')$

2. Continuity at a point

A Function f is continuous at $x = a$ if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

↳ Some continuous Functions

1. Polynomial $\forall x \in \mathbb{R}$
2. e^x $\forall x \in \mathbb{R}$
3. $\ln x$ for $x > 0$
4. Rational Function
5. $\cos x$ $\forall x \in \mathbb{R}$
6. $\sin x$ $\forall x \in \mathbb{R}$

↳ Types of discontinuities

1. point discontinuities (a hole)
2. jump discontinuities (different behavior on left/right)
3. vertical Asymptotes

↳ Extension by continuity

If f is a defined and continuous function on $]a, b[\cup]b, c[$ ($f(b)$ doesn't exist)

And if $\lim_{x \to b} f(x) = l$ ($l \in \mathbb{R}$)

then f is extendable by continuity at $x = b$ by the Fonction $g(x)$

$$g(x) = \begin{cases} f(x) & \text{if } x \neq b \\ l & \text{if } x = b \end{cases}$$

$g(x)$ is the extension of f in b

Intermediate Value Theorem

If f is a continuous function on $[a,b]$ s.t. $f(a).f(b) < 0$ ($\neq$ sign) then, $\exists\ c \in [a,b]$ such that $f(c) = 0$

If f is strictly monotone then the root (c) is unique

Squeeze Theorem

Let The functions $f(x)$, $g(x)$, and $h(x)$.

If $f(x) \leq g(x) \leq h(x)$

then if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$

$$\Rightarrow \lim_{x \to a} g(x) = l$$