Lebanese University
Faculty of Sciences
Section I

MATHEMATICS
Calculus

Solved Exams

Ibrahim ZALZALI • Haidar SEMAAN • Ziad MROUE

October 2020

Contents

Part I

Final Exams

LEBANESE UNIVERSITY — Date: 25 June 2001
Faculty of Sciences (I)

Final Exam — **BC11** — Duration: 3 hours

Exercise 1 — **[10 pts]**

Consider the double integral

$$I = \int_0^2 \mathrm{d}y \int_{\sqrt{2y}}^2 f(x, y)\,\mathrm{d}x.$$

1° Represent graphically the domain of integration D.

2° Express I in another form by reversing the order of integration.

3° Calculate $J = \iint_D \frac{y}{(1 + x^2 + y^2)^2}\,\mathrm{d}x\,\mathrm{d}y$.

Exercise 2 — **[12 pts]**

Let

$$D_1 = \left\{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 \geq 4,\ 0 \leq y \leq x \ \text{ and } \ x \leq 2\right\},$$

and, $D_2 = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 \geq 4,\ 0 \leq x \leq y \ \text{ and } \ y \leq 2\}$.

1° Represent graphically the domains D_1 and D_2.

2° Using polar coordinates, calculate the area of D_1.

3° Calculate $I = \iint_\Delta |x - y|\,\mathrm{d}x\,\mathrm{d}y$ where

$$\Delta = \left\{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 \geq 4,\ 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\right\}.$$

Exercise 3 — **[10 pts]**

1° Decompose the following rational functions into simple elements in $\mathbb{R}$:

$$\frac{1}{t^2(t^2 + 1)} \quad \text{and} \quad \frac{1}{t(t^2 + 1)}.$$

2° Calculate $\displaystyle\int \frac{\arctan e^x}{e^x(1 + e^{2x})}\,\mathrm{d}x$.

Exercise 4 ——————————————————————— **[10 pts]**

Given

$$f(x) = \begin{cases} \ln\left(\dfrac{\sqrt{1+\frac{2}{3}x^2}\ -1}{x^2}\right) & \text{if } \ x \neq 0; \\ -\ln 3 & \text{if } \ x = 0. \end{cases}$$

1° Show that f is continuous at 0.

2° Is f differentiable at 0 ? If yes, calculate $f'(0)$.

Exercise 5 ——————————————————————— **[8 pts]**

Calculate $\displaystyle \lim_{x\to+\infty} \left[\left(x^3 - x^2 + \frac{x}{2}\right)e^{\frac{1}{x}} - \sqrt{1+x^6}\right].$

Exercise 6 ——————————————————————— **[13 pts]**

Solve the following differential equations :

1° $\dfrac{\mathrm{d}y}{\mathrm{d}x} - \sqrt{y} = y.$

2° $y'' - 4y = e^{2x} + 1.$

Exercise 7 ——————————————————————— **[7 pts]**

Let $\quad f(x) = \arcsin\sqrt{\dfrac{1+\sin x}{2}} \quad$ be defined on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right].$

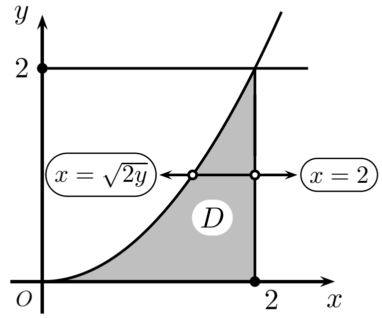
1° Calculate $f'(x)$.

2° Deduce a simple expression for $f(x)$.

Exercise 1

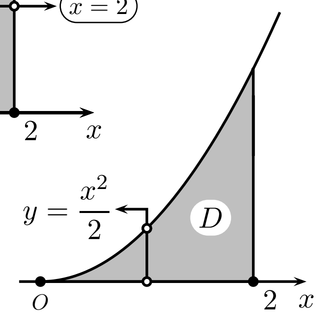
1° D is defined by

$$0 \le y \le 2;$$
$$\sqrt{2y} \le x \le 2.$$



2° D is defined by

$$0 \le x \le 2;$$
$$0 \le y \le \frac{x^2}{2}.$$



Hence, $I = \int_0^2 \mathrm{d}x \int_0^{\frac{x^2}{2}} f(x,y)\,\mathrm{d}y.$

3° $$\begin{aligned} J &= \frac{1}{2}\int_0^2 \mathrm{d}x \int_0^{\frac{x^2}{2}} \frac{(1+x^2+y^2)'_y}{(1+x^2+y^2)^2}\,\mathrm{d}y = -\frac{1}{2}\int_0^2 \left[\frac{1}{1+x^2+y^2}\right]_0^{\frac{x^2}{2}} \mathrm{d}x \\ &= -\frac{1}{2}\int_0^2 \frac{\mathrm{d}x}{1+x^2+\frac{x^4}{4}} + \frac{1}{2}\int_0^2 \frac{\mathrm{d}x}{1+x^2} \\ &= -\frac{1}{2}\int_0^2 \frac{\mathrm{d}x}{(1+x^2/2)^2} + \frac{1}{2}\arctan 2. \end{aligned}$$

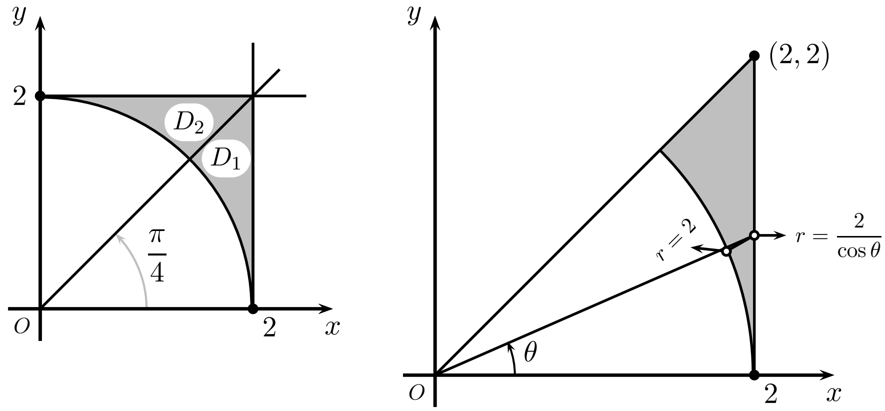
Furthermore, setting $x = \sqrt{2}\tan\theta$, it yields

$$\begin{aligned} \int \frac{\mathrm{d}x}{(1+x^2/2)^2} &= \sqrt{2}\int \frac{\mathrm{d}\theta}{1+\tan^2\theta} \\ &= \sqrt{2}\int \cos^2\theta\,\mathrm{d}\theta = \frac{\sqrt{2}}{2}\int [1+\cos(2\theta)]\,\mathrm{d}\theta \\ &= \frac{\sqrt{2}}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] = \frac{\sqrt{2}}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) \\ &= \frac{\sqrt{2}}{2}\left[\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{\sqrt{2}}\left(\frac{1}{1+\frac{x^2}{2}}\right)\right] + \text{cnst.} \end{aligned}$$

From which $\quad J = -\frac{\sqrt{2}}{4}\arctan\sqrt{2} - \frac{1}{6} + \frac{1}{2}\arctan 2.$

Exercise 2

1°



2° $$\text{Area}\,(D_1) = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_2^{\frac{2}{\cos\theta}} r\,\mathrm{d}r$$
$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_2^{\frac{2}{\cos\theta}} \mathrm{d}\theta = 2\int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} - 1\right) \mathrm{d}\theta.$$

From which, $\text{Area}\,(D_1) = 2\,[\tan\theta - \theta]_0^{\frac{\pi}{4}} = 2 - \frac{\pi}{2}.$

Question : re-find $\text{Area}\,(D_1)$ geometrically.

3° $$I = \iint_{D_1} |x-y|\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} |x-y|\,\mathrm{d}x\,\mathrm{d}y$$
$$= 2\iint_{D_1} (x-y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{(by symmetry)}.$$

From which,

$$I = 2\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_2^{\frac{2}{\cos\theta}} r(\cos\theta - \sin\theta) r\,\mathrm{d}r$$
$$= 2\int_0^{\frac{\pi}{4}} (\cos\theta - \sin\theta) \left[\frac{1}{3}r^3\right]_2^{\frac{2}{\cos\theta}} \mathrm{d}\theta$$
$$= \frac{16}{3}\int_0^{\frac{\pi}{4}} (\cos\theta - \sin\theta) \left(\frac{1}{\cos^3\theta} - 1\right) \mathrm{d}\theta$$
$$= \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\cos^2\theta} - \frac{16}{3}\int_0^{\frac{\pi}{4}} \frac{\sin\theta\,\mathrm{d}\theta}{\cos^3\theta} - \frac{16}{3}\int_0^{\frac{\pi}{4}} (\cos\theta - \sin\theta)\,\mathrm{d}\theta$$
$$= 8 - \frac{16}{3}\sqrt{2}.$$

Exercise 3

1° Decomposing into simple rational elements as follows

$$\frac{1}{t^2(1+t^2)} = \frac{A}{t^2} + \frac{B}{1+t^2} \quad ; \quad \frac{1}{t(1+t^2)} = \frac{A'}{t} + \frac{B't+C'}{1+t^2}.$$

The constants are found to be : $A = 1$, $B = -1$, $A' = 1$, $B' = -1$, and $C' = 0$.

2° Set $t = e^x$;

$$I(x) = \int \frac{\arctan e^x}{e^x(1+e^{2x})}\,\mathrm{d}x = \int \frac{\arctan t}{t^2(1+t^2)}\,\mathrm{d}t$$

Using the above decomposition and integrating by parts, we have

$$u = \arctan t, \qquad \text{so} \quad \mathrm{d}u = \frac{\mathrm{d}t}{1+t^2};$$

$$\mathrm{d}v = \frac{\mathrm{d}t}{t^2(1+t^2)}, \quad \text{so} \quad v = -\frac{1}{t} - \arctan t,$$

it yields,

$$\int \frac{\arctan t}{t^2(1+t^2)}\,\mathrm{d}t = -\frac{\arctan t}{t} - \arctan^2 t + \int \left(\frac{1}{t} + \arctan t\right) \frac{\mathrm{d}t}{1+t^2}.$$

Thus $\quad I(x) = -\dfrac{1}{2}\left[(\arctan e^x)^2 + \ln\left(1+e^{2x}\right)\right] - \dfrac{\arctan e^x}{e^x} + x + \text{cnst}.$

Exercise 4

1° In the neighborhood of $x = 0$, we have

$$\frac{\sqrt{1+\frac{2}{3}x^2} - 1}{x^2} = \frac{1}{3} + x\varepsilon(x),$$

with $\lim\limits_{x\to 0} \varepsilon(x) = 0$.

$\lim\limits_{x\to 0} f(x) = \ln \dfrac{1}{3} = f(0)$, hence f is continuous at 0.

2° On the other hand,

$$\frac{f(x) - f(0)}{x} = \frac{\ln\left(\frac{1}{3} + x\varepsilon(x)\right) + \ln 3}{x} = \frac{\ln(1 + x\varepsilon(x))}{x}.$$

Thus,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{\ln(1 + x\varepsilon(x))}{x} = \lim_{x\to 0} \varepsilon(x) = 0.$$

Hence f is differentiable at 0, and $f'(0) = 0$.

Exercise 5

Set $f(x) = \left(x^3 - x^2 + \frac{x}{2}\right) e^{\frac{1}{x}} - \sqrt{1+x^6}$, and $t = \frac{1}{x}$. The finite expansion of $f\left(\frac{1}{t}\right)$ to order 0 near $t = 0_+$, is

$$\begin{aligned} f\left(\frac{1}{t}\right) &= \frac{\left(1 - t + \frac{t^2}{2}\right) e^t - \sqrt{1+t^6}}{t^3} \\ &= \frac{\left(1 - t + \frac{t^2}{2}\right)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right) - 1}{t^3} + \varepsilon(t) = \frac{1}{6} + \varepsilon(t) \end{aligned}$$

with $\lim_{t\to 0} \varepsilon(t) = 0$. Therefore, $\lim_{x\to+\infty} f(x) = \lim_{t\to 0_+} f\left(\frac{1}{t}\right) = \frac{1}{6}$.

Exercise 6

1° The given differential equation is rewritten as : $\frac{\mathrm{d}y}{\sqrt{y}\,(1+\sqrt{y})} = \mathrm{d}x$.

From which $2\ln(1+\sqrt{y}) = x + c$, so $y(x) = \left(ke^{\frac{x}{2}} - 1\right)^2$.

2° The characteristic equation $r^2 - 4 = 0$ admits two roots -2 and 2. Thus the general solution of the associated homogeneous equation is :

$$y_1(x) = c_1 e^{-2x} + c_2 e^{2x}.$$

- A particular solution of $y'' - 4y = e^{2x}$, which is of the form $Y_1(x) = \alpha x e^{2x}$ can be easily obtained as $Y_1(x) = \frac{1}{4} x e^{2x}$.
- A particular solution of $y'' - 4y = 1$, which is of the form $Y_2(x) = k$ can be easily obtained as $Y_2(x) = -\frac{1}{4}$.

Thus finally, $y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{4} + \frac{1}{4} x e^{2x}$.

Exercise 7

1° We have $y' = \frac{\left(\sqrt{\frac{1+\sin x}{2}}\right)'}{\sqrt{1 - \frac{1+\sin x}{2}}} = \frac{1}{2}\frac{\cos x}{|\cos x|} = \frac{1}{2}$, since $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

2° By integration, we get $y = \frac{x}{2} + C$.

For $x = 0$, $C = y(0) = \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $y = x + \frac{\pi}{4}$.

LEBANESE UNIVERSITY Date: 01 July 2002
Faculty of Sciences (I)

Final Exam **BC11** Duration: 3 hours

Exercise 1

Calculate the following integrals :

$$I_1 = \int \frac{(1+x)^2}{2+x^2}\,\mathrm{d}x \; ; \quad I_2 = \int_1^{\sqrt{3}} \frac{\arcsin^2 \frac{x}{2}}{\sqrt{4-x^2}}\,\mathrm{d}x \; ; \quad I_3 = \int \frac{1}{\sin^2 x \cos^2 x}\,\mathrm{d}x.$$

Exercise 2

1° Calculate $I_1 = \displaystyle\int \frac{\mathrm{d}t}{\sin t}$.

2° Calculate $I_2 = \displaystyle\int \frac{t\cos t}{\sin^2 t}\,\mathrm{d}t$.

Exercise 3

Evaluate $\displaystyle\lim_{x\to 0}\left(\frac{x}{\sin x - x} + \frac{x}{\sinh x - x}\right)$.

Exercise 4

1° Give the finite expansion of order 3 of $\arctan \dfrac{1+x}{1-x}$ in the neighborhood of $x = 0$.

2° Determine the asymptote (Δ) at $+\infty$ to the representative curve (C) of the function f defined by $f(x) = x \arctan \dfrac{x+1}{x-1}$. Precise the relative positions of (C) and (Δ) at $+\infty$.

Exercise 5

Consider the double integral $I = \displaystyle\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where f is a continuous function over a domain D defined by :

$$D = \{(x,y) \in \mathbb{R}^2 \,;\, 0 \le x \le 1,\ (x-1)^2 + y^2 \ge 1,\ \text{and } y \le (x-1)^2 + 1\}$$

1° Sketch the domain of integration D.

2° Using cartesian coordinates, write down I, as a succession of two simple iterative integrals, in two different forms.

3° Calculate the integral I in case $f(x,y) = 2y$.

Exercise 6

Given the double integral

$$I = \int_0^1 \mathrm{d}y \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} \frac{xy}{x^2+y^2}\,\mathrm{d}x.$$

1° Sketch the domain of integration D.

2° Calculate I by using polar coordinates.

Exercise 7

Solve the following first order differential equation :

$$xy' = 2y + y(\ln y - \ln x)^2.$$

Exercise 8

Let (E) be the following second order differential equation with constant coefficients :

$$y'' + 2y' + y = 2\cos^2 x \qquad (E)$$

1° Write down the associated to (E) homogeneous equation (H), and give its general solution.

2° (a) Find a particular solution of the differential equation

$$y'' + 2y' + y = 1 \qquad (E_1)$$

(b) Find a particular solution of the differential equation

$$y'' + 2y' + y = \cos(2x) \qquad (E_2)$$

(c) Deduce a particular solution of the differential equation (E).

(d) Give the general solution of (E).

Exercise 1

Evaluation of I_1. By Euclidian division of $(x+1)^2 = x^2+2x+1$ by x^2+2, we easily obtain

$$\begin{aligned} I_1 &= \int \left(1 + \frac{2x}{2+x^2} - \frac{1}{2+x^2}\right) \mathrm{d}x \\ &= x + \ln\left(2+x^2\right) - \frac{\sqrt{2}}{2} \arctan\left(\frac{\sqrt{2}}{2} x\right) + c, \text{ with } c \in \mathbb{R}. \end{aligned}$$

Evaluation of I_2. Remark that $\left(\arcsin \frac{x}{2}\right)' = \frac{1}{\sqrt{4-x^2}}$. Thus set $t = \arcsin \frac{x}{2}$. It follows that

$$I_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} t^2 \,\mathrm{d}t = \left[\frac{t^3}{3}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{7}{648}\pi^3.$$

Evaluation of I_3.

First method :

$$I_3 = \int \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} \,\mathrm{d}x = \int \frac{\mathrm{d}x}{\cos^2 x} + \int \frac{\mathrm{d}x}{\sin^2 x} = \tan x - \cot x + c, \text{ with } c \in \mathbb{R}.$$

Second method :

Let $\tan x = t$, hence $\mathrm{d}t = \frac{\mathrm{d}x}{\cos^2 x}$ and $\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$. It follows that

$$I_3 = \int \frac{1}{\sin^2 x} \cdot \frac{\mathrm{d}x}{\cos^2 x} = \int \frac{1+t^2}{t^2} \,\mathrm{d}t = \int \left(\frac{1}{t^2} + 1\right) \mathrm{d}t = -\frac{1}{t} + t + c = \tan x - \cot x + c.$$

Third method :

Integrating by parts with $u = \frac{1}{\sin^2 x}$ and $\mathrm{d}v = \frac{\mathrm{d}x}{\cos^2 x}$.
Hence, $\mathrm{d}u = -2\frac{\cos x}{\sin^3 x} \,\mathrm{d}x$ and $v = \tan x$. Therefore,

$$I_3 = \frac{\tan x}{\sin^2 x} + 2\int \tan x \frac{\cos x \,\mathrm{d}x}{\sin^3 x} = \frac{1}{\sin x \cos x} + 2\int \frac{\mathrm{d}x}{\sin^2 x} = \frac{1}{\sin x \cos x} - 2\cot x + c.$$

Fourth method : By virtue of $\cos^2 x \sin^2 x = \frac{1}{4} \sin^2(2x)$, thus

$$I_3 = \int \frac{4}{\sin^2(2x)} \,\mathrm{d}x = -2\cot(2x) + c.$$

Exercise 2

1° $I_1 = \int \frac{\mathrm{d}t}{\sin t} = \int \frac{\sin t \,\mathrm{d}t}{\sin^2 t} = \int \frac{\sin t \,\mathrm{d}t}{1-\cos^2 t} = -\frac{1}{2} \ln \frac{1+\cos t}{1-\cos t} + c$, with $c \in \mathbb{R}$.

2° Performing an integration by parts, we set

$$u = t, \quad \text{hence} \quad \mathrm{d}u = \mathrm{d}t$$

$$\mathrm{d}v = \frac{\cos t}{\sin^2 t}\mathrm{d}t, \quad \text{hence} \quad v = -\frac{1}{\sin t}.$$

Therefore,

$$I_2 = -\frac{t}{\sin t} + \int \frac{\mathrm{d}t}{\sin t},$$

from which, due to 1° : $I_2 = -\frac{t}{\sin t} - \frac{1}{2} \ln \frac{1+\cos t}{1-\cos t} + c$, where $c \in \mathbb{R}$.

Exercise 3

$$\begin{aligned}\lim_{x\to 0} \left(\frac{x}{\sin x - x} + \frac{x}{\sinh x - x}\right) &= \lim_{x\to 0} \frac{x(\sinh x - x) + x(\sin x - x)}{(\sin x - x)(\sinh x - x)} \\ &= \lim_{x\to 0} \frac{x\left(\frac{x^3}{3!} + \frac{x^5}{5!}\right) + x\left(-\frac{x^3}{3!} + \frac{x^5}{5!}\right)}{\left(-\frac{x^3}{3!}\right)\left(\frac{x^3}{3!}\right)} = -\frac{3}{5}.\end{aligned}$$

Exercise 4

1° We have, for $x \neq 1$, $\left(\arctan \frac{1+x}{1-x}\right)' = \frac{1}{1+x^2} = 1 - x^2 + x^2\varepsilon(x)$.
Hence in neighborhood of $x = 0$, and to order 3, we have

$$\arctan \frac{1+x}{1-x} = \arctan \frac{1+0}{1-0} + x - \frac{x^3}{3} + x^3\varepsilon(x).$$

Therefore, $\arctan \frac{1+x}{1-x} = \frac{\pi}{4} + x - \frac{x^3}{3} + x^3\varepsilon(x)$, with $\lim_{x\to 0} \varepsilon(x) = 0$.

2° Set $t = \frac{1}{x}$. It yields $tf\left(\frac{1}{t}\right) = \arctan \frac{1+t}{1-t}$. Thus, due to 1° :

$$tf\left(\frac{1}{t}\right) = \frac{\pi}{4} + t - \frac{t^3}{3} + t^3\varepsilon(t), \quad \text{with} \quad \lim_{t\to 0} \varepsilon(t) = 0.$$

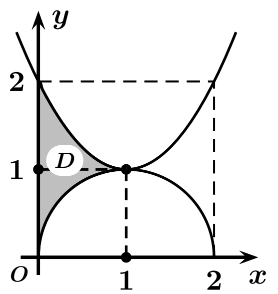
Hence,

$$f(x) = 1 + \frac{\pi}{4}x - \frac{1}{3x^2} + \frac{1}{x^2}\varepsilon(x).$$

Thus the equation of the asymptote (Δ) is $y = 1 + \frac{\pi}{4}x$. On the other hand, $f(x) - y \underset{+\infty}{\simeq} -\frac{1}{3x^2} < 0$, so (Δ) is situated above (C).

Exercise 5

1° D is represented in the adjacent figure.

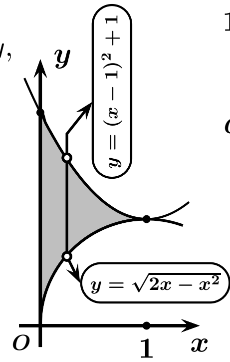


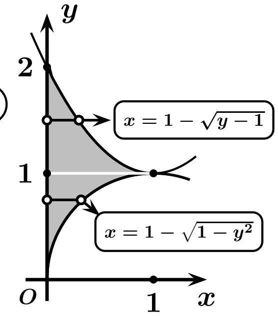
2° We have

$$I = \int_0^1 \mathrm{d}x \int_{\sqrt{2x-x^2}}^{(x-1)^2+1} f(x,y)\,\mathrm{d}y,$$

or

$$I = \int_0^1 \mathrm{d}y \int_0^{1-\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_0^{1-\sqrt{y-1}} f(x,y)\,\mathrm{d}x.$$



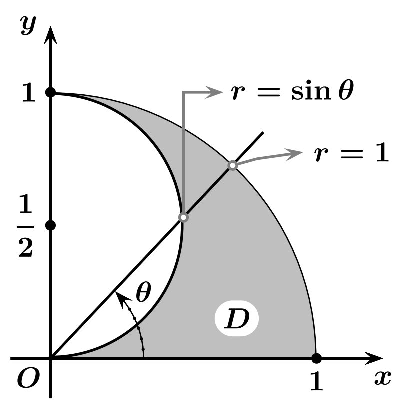


3° $$I = \int_0^1 \mathrm{d}x \int_{\sqrt{2x-x^2}}^{(x-1)^2+1} 2y\,\mathrm{d}y$$

$$= \int_0^1 (x^4 - 4x^3 + 9x^2 - 10x + 4)\,\mathrm{d}x = \frac{6}{5}.$$

Exercise 6

1° The domain D is bounded by the curves of equations :
$y = 0$, $y = 1$, $x = \sqrt{y - y^2}$ (i.e. semi circle $x^2 + (y - \frac{1}{2})^2 = 1$, $x \geq 0$) and $x = \sqrt{1 - y^2}$ (i.e. quarter circle $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$).
Thus D is sketched as shown.



2° Using polar coordinates, the domain D is defined as

$$D : \begin{cases} 0 \leq \theta \leq \dfrac{\pi}{2}; \\ \sin\theta \leq r \leq 1. \end{cases}$$

From which,

$$I = \iint_{\Delta} \frac{r\cos\theta \times r\sin\theta}{r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\,\mathrm{d}\theta \int_{\sin\theta}^1 r\,\mathrm{d}r$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta(1 - \sin^2\theta)\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos^3\theta\,\mathrm{d}\theta = -\frac{1}{2}\int_1^0 t^3\,\mathrm{d}t = \frac{1}{8},$$

where the change of variable $t = \cos\theta$ had been performed.

Exercise 7

This is a homogeneous differential equation, since

$$y' = 2\frac{y}{x} + \frac{y}{x}\left(\ln\frac{y}{x}\right)^2.$$

Let $z = \dfrac{y}{x}$. It follows that :

$$xz' + z = 2z + z\ln^2 z \quad \Rightarrow \quad \frac{\frac{\mathrm{d}z}{z}}{1+\ln^2 z} = \frac{\mathrm{d}x}{x} \quad \Rightarrow \quad \frac{\mathrm{d}(\ln z)}{1+\ln^2 z} = \frac{\mathrm{d}x}{x}.$$

Integrating, we obtain : $\arctan\left(\ln\dfrac{y}{x}\right) = \ln x + c$, where $c \in \mathbb{R}$.

Exercise 8

1° The homogeneous equation (H), associated to (E), is

$$y'' + 2y' + y = 0.$$

Its characteristic equation $r^2+2r+1=0$ admits double roots $r=-1$. Hence, the general solution of (H) is : $y_H = c_1e^{-x} + c_2xe^{-x}$.

2° (a) Seeking a particular solution of (E_1) in the form $y_{p_1} = a$. Upon substitution, we obtain $a = 1$.

(b) Seeking a particular solution of (E_2) in the form $y_{p_2} = a\cos(2x)+b\sin(2x)$. Upon substitution, we obtain $a = -\dfrac{3}{25}$ and $b = \dfrac{4}{25}$.

(c) A particular solution of (E) would be

$$y_{p_1} + y_{p_2} = 1 - \frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x).$$

Note that $2\cos^2 x = 1 + \cos(2x)$.

(d) The general solution of (E) is :

$$y\left(x\right) = c_1e^{-x} + c_2xe^{-x} + 1 - \frac{3}{25}\cos(2x) + \frac{4}{25}\sin(2x).$$

LEBANESE UNIVERSITY Date: 17 June 2003

Faculty of Sciences (I)

Final Exam **BC11** Duration: 3 hours

ANSWER PAGE 21

Exercise 1 **[5 pts]**

1° Solve the differential equation

$$y' = \frac{y-x}{y+x} \qquad (E)$$

2° Determine the solution y of (E), verifying $y(1) = 0$.

Exercise 2 **[10 pts]**

Consider the following second order differential equation (E) with constant coefficients

$$y'' + y' - 2y = 10\sin x + 6e^x \qquad (E)$$

1° Determine the general solution of (E).

2° Determine the solution y of (E), verifying $y(0) = 2$ and $y'(0) = 1$.

Exercise 3 **[8 pts]**

Calculate the following integrals :

$$I_1 = \int \frac{2x-1}{x(x-1)}\,\mathrm{d}x\ ; \quad I_2 = \int_3^8 \frac{\sqrt{x+1}}{x}\,\mathrm{d}x\ ; \quad I_3 = \int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x.$$

Exercise 4 **[8 pts]**

Calculate the following integrals :

1° $I_1 = \displaystyle\int_1^{\frac{1}{\sqrt{3}}} \frac{1}{x^2} \arctan\frac{1}{x}\,\mathrm{d}x.$

2° $I_2 = \displaystyle\int \cos x \ln(\tan x)\,\mathrm{d}x.$

Exercise 5 ________________________________ **[10 pts]**

1° Give the finite expansion of order 4, in neighborhood of $x = 0$, of :

$$\ln(\cos 2x); \qquad \frac{2x^2}{1+x}; \qquad (\sin 2x)^2;$$

further, of : $f(x) = 3\ln(\cos 2x) + \dfrac{2x^2}{1+x} + (\sin 2x)^2$.

2° Deduce the limit of $\dfrac{f(x) + 2x^3}{x^4}$ as x tends to 0.

Exercise 6 ________________________________ **[8 pts]**

Consider the function f, defined on $\mathbb{R}$, by :

$$f(x) = \frac{\pi}{2}x - x\arctan x.$$

Determine the equation of the asymptote to the curve of f at $+\infty$, and then precise their relative positions in neighborhood of $+\infty$.

Exercise 7 ________________________________ **[14 pts]**

Given the double integral $I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where f is a continuous function over the domain D, defined as :

$$D = \{(x,y) \in \mathbb{R}^2 \ ; \ \ 0 \le y, \ \ x^2 + y^2 - x \le 0, \ \ \text{and } x^2 + y^2 - y \ge 0\}.$$

1° Sketch the domain of integration D.

2° Use cartesian coordinates to write down I in two distinct forms, by altering the order of integration.

3° Using polar coordinates, calculate the integral I in the case when $f(x,y) = x^2 + y^2$.

Exercise 8 ________________________________ **[12 pts]**

Given the following double integral

$$I = \int_0^1 \mathrm{d}x \int_{\sqrt{x}}^1 \cos\left(\frac{x}{y}\right) \mathrm{d}y.$$

1° Sketch the domain of integration D.

2° Evaluate I.

Final exam Year 2002-2003	Solution	Calculus BC11

Exercise 1

1° It is the case of a homogeneous differential equation since $y' = \dfrac{\dfrac{y}{x} - 1}{\dfrac{y}{x} + 1}$.

Set $z = \dfrac{y}{x}$. It follows that

$$xz' + z = \frac{z-1}{z+1} \quad \Leftrightarrow \quad \frac{1+z}{1+z^2}\,\mathrm{d}z = -\frac{\mathrm{d}x}{x}.$$

Hence, $\quad \arctan\dfrac{y}{x} + \ln\sqrt{1+\left(\dfrac{y}{x}\right)^2} = -\ln|x| + c, \quad \forall c \in \mathbb{R}.$

2° $y(1) = 0$ yields $c = 0$.

Exercise 2

1° The homogeneous equation (H) associated to (E) is :

$$y'' + y' - 2y = 0.$$

The associated characteristic equation $r^2 + r - 2 = 0$ admits the roots $r_1 = 1$ and $r_2 = -2$. Therefore, the general solution of (H) is :

$$y_H = c_1 e^{-2x} + c_2 e^x.$$

We look for a particular solution of $(E_1) : y'' + y' - 2y = 10\sin x$ in the form $y_{p_1} = a\cos x + b\sin x$. By substitution, we obtain $a = -1$ and $b = -3$.
And looking for a particular solution of $(E_2) : y'' + y' - 2y = 6e^x$ in the form $y_{p_2} = axe^x$. By substitution, we obtain $a = 2$.
Thus the general solution of (E) is :

$$y(x) = c_1 e^x + c_2 e^{-2x} - \cos x - 3\sin x + 2xe^x.$$

2° Whence $y(0) = 2$ and $y'(0) = 1$, we get

$$\begin{cases} c_1 + c_2 = 3, \\ c_1 - 2c_2 = 2. \end{cases}$$

From which follows that $c_1 = \dfrac{8}{3}$ and $c_2 = \dfrac{1}{3}$.
Thus finally, $y(x) = \left(2x + \dfrac{8}{3}\right)e^x + \dfrac{1}{3}e^{-2x} - \cos x - 3\sin x$.

Exercise 3

Evaluation of I_1. Let $t = x(x-1)$, it implies

$$I_1 = \int \frac{\mathrm{d}t}{t} = \ln|t| + \text{cnst} = \ln|x(x-1)| + \text{cnst}.$$

Evaluation of I_2. Set $\sqrt{x+1} = t$, it implies

$$I_2 = 2\int_2^3 \frac{t^2}{t^2-1}\,\mathrm{d}t = 2\int_2^3 \left(1 - \frac{1}{1-t^2}\right)\mathrm{d}t = \left[2t - \ln\left|\frac{1+t}{1-t}\right|\right]_2^3 = 2 + \ln\frac{3}{2}.$$

Evaluation of I_3.

$$\begin{aligned} I_3 &= \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos(2x)}{2}\right)^2 \mathrm{d}x = \frac{1}{4}\int_0^{\frac{\pi}{2}} \Big(1 + 2\cos(2x) + \cos^2(2x)\Big)\,\mathrm{d}x \\ &= \frac{1}{4}\int_0^{\frac{\pi}{2}} \left(1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}\right)\mathrm{d}x \\ &= \left[\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right]_0^{\frac{\pi}{2}} = \frac{3}{16}\pi. \end{aligned}$$

Exercise 4

1° Set $t = \dfrac{1}{x}$. Hence,

$$I_1 = -\int_1^{\sqrt{3}} \arctan t\,\mathrm{d}t.$$

Performing an integration by parts (with $u = \arctan t$ and $\mathrm{d}v = \mathrm{d}t$). It follows that

$$\begin{aligned} I_1 &= -\left[t\arctan t\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{t\,\mathrm{d}t}{1+t^2} \\ &= -\frac{\sqrt{3}}{3}\pi + \frac{\pi}{4} + \frac{1}{2}\left[\ln(1+t^2)\right]_1^{\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3}\pi + \frac{\pi}{4} + \frac{1}{2}\ln 2. \end{aligned}$$

2° Integrating by parts (with $u = \ln(\tan x)$ and $\mathrm{d}v = \cos x\,\mathrm{d}x$), we get

$$\begin{aligned} I_2 &= \sin x\ln(\tan x) - \int \frac{\mathrm{d}x}{\cos x} \\ &= \sin x\ln(\tan x) - \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + c, \qquad \forall c \in \mathbb{R}. \end{aligned}$$

Exercise 5

1° Obviously, in the neighborhood of $x=0$ we have

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + x^4\varepsilon(x) = 1 - 2x^2 + \frac{2}{3}x^4 + x^4\varepsilon(x)$$

$$\begin{aligned} \text{and,}\quad \ln(\cos 2x) &= \ln\left[1 + \left(-2x^2 + \frac{2}{3}x^4\right)\right] + x^4\varepsilon(x) \\ &= \left(-2x^2 + \frac{2}{3}x^4\right) - \frac{1}{2}\left(-2x^2 + \frac{2}{3}x^4\right)^2 + x^4\varepsilon(x) \\ &= -2x^2 - \frac{4}{3}x^4 + x^4\varepsilon(x). \end{aligned}$$

On the other hand,

$$\frac{2x^2}{1+x} = 2x^2\left(1 - x + x^2\right) + x^4\varepsilon(x) = 2x^2 - 2x^3 + 2x^4 + x^4\varepsilon(x)$$

$$\text{and,}\quad \sin^2(2x) = \left(2x - \frac{(2x)^3}{3!}\right)^2 + x^4\varepsilon(x) = 4x^2 - \frac{16}{3}x^4 + x^4\varepsilon(x).$$

That implies $f(x) = -2x^3 - \dfrac{22}{3}x^4 + x^4\varepsilon(x)$, with $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

2° $\dfrac{f(x)+2x^3}{x^4} \underset{0}{\simeq} -\dfrac{22}{3} + \varepsilon(x)$. Hence $\lim\limits_{x\to 0}\dfrac{f(x)+2x^3}{x^4} = -\dfrac{22}{3}$.

Exercise 6

Set $t = \dfrac{1}{x}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. We give the f.e. of $tf\left(\dfrac{1}{t}\right)$ to order 3 in neighborhood of $t=0$. We have

$$tf\left(\frac{1}{t}\right) = \frac{\pi}{2} - \arctan\frac{1}{t}.$$

We have

$$\left(\arctan\frac{1}{t}\right)' = -\frac{1}{1+t^2} = -1 + t^2 + t^2\varepsilon(t).$$

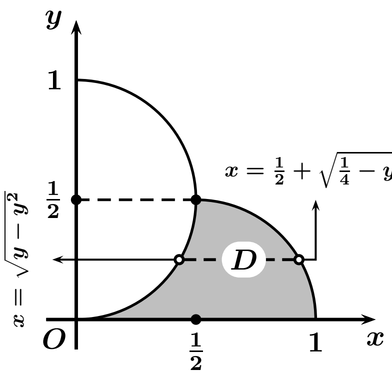
Hence, $\arctan\dfrac{1}{t} = \dfrac{\pi}{2} - t + \dfrac{t^3}{3} + t^3\varepsilon(t)$. Thus $\quad tf\left(\dfrac{1}{t}\right) = t - \dfrac{t^3}{3} + t^3\varepsilon(t)$. It follows that

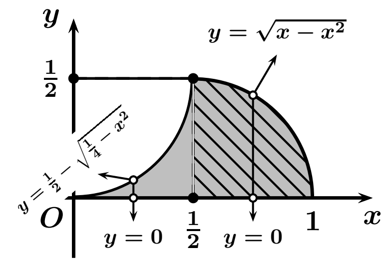
$$f(x) = 1 - \frac{1}{3x^2} + \frac{1}{x^2}\varepsilon(x), \qquad \lim_{x\to+\infty}\varepsilon(x) = 0.$$

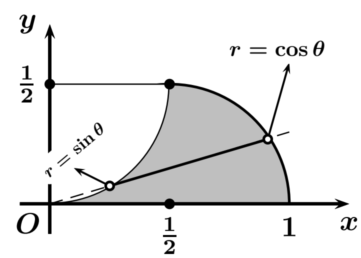
The equation of the asymptote is $y=1$, and it is situated above the curve.

Exercise 7

1° Graphical representation of D :







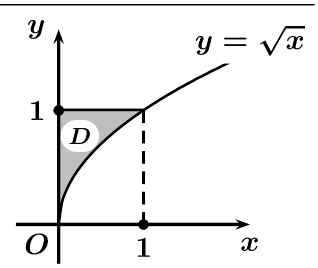
2° $I = \displaystyle\int_0^{\frac{1}{2}} \mathrm{d}y \int_{\sqrt{y-y^2}}^{\frac{1+\sqrt{1-4y^2}}{2}} f(x,y)\,\mathrm{d}x$

or, $I = \displaystyle\int_0^{\frac{1}{2}} \mathrm{d}x \int_0^{\frac{1-\sqrt{1-4x^2}}{2}} f(x,y)\,\mathrm{d}y + \int_{\frac{1}{2}}^{1} \mathrm{d}x \int_0^{\sqrt{x-x^2}} f(x,y)\,\mathrm{d}y.$

3° $$I = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\sin\theta}^{\cos\theta} r^3\,\mathrm{d}r = \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\cos^4\theta - \sin^4\theta\right)\mathrm{d}\theta$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(\cos^2\theta - \sin^2\theta\right)\mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{\pi}{4}} \cos 2\theta\,\mathrm{d}\theta = \frac{1}{4}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{8}.$$

Exercise 8

1° The domain of integration D is bounded by the curves, of equations $x = 0$, $x = 1$, $y = \sqrt{x}$ and $y = 1$. Thus the graphical representation of D is shown as :



2° Holding y fixed, D is defined by :

$$\begin{cases} 0 \le y \le 1; \\ 0 \le x \le y^2. \end{cases}$$

Thus $$I = \int_0^1 \mathrm{d}y \int_0^{y^2} \cos\left(\frac{x}{y}\right)\mathrm{d}x = \int_0^1 \left[y\sin\left(\frac{x}{y}\right)\right]_{x=0}^{x=y^2} \mathrm{d}y$$
$$= \int_0^1 y\sin y\,\mathrm{d}y \overset{\text{by parts}}{=} [\sin y - y\cos y]_0^1 = \sin 1 - \cos 1.$$

LEBANESE UNIVERSITY Date: 17-07-2004
Faculty of Sciences (I)

Final I-session **BC11** **Duration: 3 hours**

ANSWER PAGE 27

Exercise 1 **[15 pts]**

1° Evaluate the following integrals :

$$I_1 = \int_2^4 \frac{\ln x}{\sqrt{x-1}}\,\mathrm{d}x\,;$$

$$I_2 = \int_0^1 \arctan\sqrt{1-x^2}\,\mathrm{d}x\,;$$

$$I_3 = \int_0^1 \frac{x^3+x^2+1}{(x^2+1)^2}\,\mathrm{d}x.$$

2° (a) Calculate $J = \displaystyle\int_0^1 \frac{t}{(1+t)^2}\,\mathrm{d}t$.

(b) Deduce the value of $K = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin x}{(\sin x+\cos x)^2\cos x}\,\mathrm{d}x$.

Exercise 2 **[15 pts]**

1° Give the f.e. of order 3 of $\ln(1+\tan x)$ in neighborhood of 0.

2° (a) Give the f.e. of order 3 of $\arctan\dfrac{1+x}{1-x}$ in neighborhood of 0.

(b) Deduce the f.e. of order 3 of $\cos\left(2\arctan\dfrac{1+x}{1-x}\right)$ near 0.

3° Determine $\displaystyle\lim_{x\to 0}\frac{\ln(1+\tan x)+x+\dfrac{x^2}{2}+\cos\left(2\arctan\dfrac{1+x}{1-x}\right)}{x^3}$.

Exercise 3 **[10 pts]**

Determine the equation of the oblique asymptote (Δ) at $+\infty$ to the representative curve (C) of the function f, defined by :

$$f(x) = e^{-\frac{1}{x+2}}\sqrt{x^2+2x},$$

and then, precise the relative position of (C) and (Δ) at $+\infty$.

Exercise 4 **[13 pts]**

Let D be a domain of the plane $\mathbb{R}^2$, defined by :

$$D = \left\{ (x, y) \in \mathbb{R}^2 \ : \ 0 \leq y \leq \frac{1}{2} \ \text{ and } \ \sqrt{2y - y^2} \leq x \leq 1 + \sqrt{1 - y^2} \right\}.$$

1° Sketch D.

2° Write down the expression of $I = \iint_D f(x, y)\, \mathrm{d}x\, \mathrm{d}y$ in two different forms, by using cartesian coordinates.

Exercise 5 **[12 pts]**

Let $I = \displaystyle\int_{-2}^{-1} \mathrm{d}x \int_{x+2}^{\sqrt{-2x-x^2}} \frac{x + y}{x^2 + y^2}\, \mathrm{d}y.$

1° Sketch the domain of integration D.

2° Using polar coordinates, calculate I.

Exercise 6 **[10 pts]**

1° Solve the differential equation

$$xy' - y = x \tan \frac{y}{x}.$$

Find the particular solution that verifies $y(1) = \dfrac{\pi}{2}$.

2° Solve the differential equation

$$(1 + x)y' + \frac{1}{x}y = \frac{(1 + x)^2}{x^3}.$$

Find the particular solution that verifies $y(1) = 1$.

Exercise 1

1° **Evaluation of I_1** Integrating by parts, we set

$$\begin{cases} u = \ln x \\ \mathrm{d}v = \dfrac{\mathrm{d}x}{\sqrt{x-1}} \end{cases} \Rightarrow \begin{cases} \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \\ v = 2\sqrt{x-1} \end{cases}$$

The formula of the integration by parts, permits writing

$$\begin{aligned} I_1 &= \left[2\sqrt{x-1}\ln x\right]_2^4 - 2\int_2^4 \frac{\sqrt{x-1}}{x}\,\mathrm{d}x \\ &= 2\left(2\sqrt{3}-1\right)\ln 2 - 2\int_2^4 \frac{\sqrt{x-1}}{x}\,\mathrm{d}x. \end{aligned}$$

With the change of variable $\sqrt{x-1} = t$, we obtain

$$\begin{aligned} \int_2^4 \frac{\sqrt{x-1}}{x}\,\mathrm{d}x &= 2\int_1^{\sqrt{3}} \frac{t^2}{1+t^2}\,\mathrm{d}t = 2\int_1^{\sqrt{3}} \left(1 - \frac{1}{1+t^2}\right)\mathrm{d}t \\ &= 2(\sqrt{3}-1) - 2\left[\arctan t\right]_1^{\sqrt{3}} = 2(\sqrt{3}-1) - \frac{\pi}{6}. \end{aligned}$$

Hence, $I_1 = 4\sqrt{3}\ln 2 - 4\sqrt{3} + \dfrac{\pi}{3} - 2\ln 2 + 4$.

Evaluation of I_2 Integrating by parts, where

$$\begin{cases} u = \arctan\sqrt{1-x^2} \\ \mathrm{d}v = \mathrm{d}x \end{cases} \text{ thus } \begin{cases} \mathrm{d}u = \dfrac{-x\,\mathrm{d}x}{(2-x^2)\sqrt{1-x^2}} \\ v = x \end{cases}$$

The formula of the integration by parts, permits writing

$$I_2 = \underbrace{\left[x\arctan\sqrt{1-x^2}\right]_0^1}_{=0} + \int_0^1 \frac{x^2\,\mathrm{d}x}{(2-x^2)\sqrt{1-x^2}}.$$

Setting $x = \sin\theta$, we get

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\,\mathrm{d}\theta}{2-\sin^2\theta} = \int_0^{\frac{\pi}{2}} \frac{1-\cos^2\theta}{1+\cos^2\theta}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{2\,\mathrm{d}\theta}{1+\cos^2\theta} - \int_0^{\frac{\pi}{2}} \mathrm{d}\theta.$$

That yields the result

$$I_2 = \int_0^{\frac{\pi}{2}} \frac{2\,\mathrm{d}(\tan\theta)}{2+\tan^2\theta} - \frac{\pi}{2} = \frac{2}{\sqrt{2}}\left[\arctan\frac{\tan\theta}{\sqrt{2}}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2} = \frac{\pi}{\sqrt{2}} - \frac{\pi}{2}.$$

Evaluation of I_3 We may write $I_3 = \int_0^1 \left(\frac{1}{1+x^2} + \frac{x^3}{(x^2+1)^2} \right) \mathrm{d}x$.

On the other hand, we have

$$\int_0^1 \frac{\mathrm{d}x}{1+x^2} = \Big[\arctan x\Big]_0^1 = \frac{\pi}{4}.$$

Also

$$\int_0^1 \frac{x^3\,\mathrm{d}x}{(1+x^2)^2} = \int_0^1 \frac{x^2 x\,\mathrm{d}x}{(1+x^2)^2},$$

so by changing the variable $t = 1 + x^2$, we obtain

$$\int_0^1 \frac{x^3\,\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\int_1^2 \frac{(t-1)\,\mathrm{d}t}{t^2} = \frac{1}{2}\left[\ln t + \frac{1}{t}\right]_1^2 = \frac{1}{2}\left(\ln 2 - \frac{1}{2}\right).$$

Finally, $I_3 = -\frac{1}{4} + \frac{1}{2}\ln 2 + \frac{\pi}{4}$.

2° (a) $J = \int_0^1 \frac{t+1-1}{(1+t)^2}\,\mathrm{d}t = \int_0^1 \left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right] \mathrm{d}t$. Hence,

$$J = \left[\ln(1+t) + \frac{1}{1+t}\right]_0^1 = \ln 2 - \frac{1}{2}.$$

(b) $K = \int_0^{\frac{\pi}{4}} \frac{\tan x}{(\tan x + 1)^2} \frac{\mathrm{d}x}{\cos^2 x} \underset{t=\tan x}{=} \int_0^1 \frac{t}{(1+t)^2}\,\mathrm{d}t = J.$

Exercise 2

1° In the neighborhood of $x = 0$, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{3!}x^3}{1 - \frac{1}{2!}x^2} + x^3\varepsilon(x) = x + \frac{1}{3}x^3 + x^3\varepsilon(x).$$

Thus,

$$\begin{aligned}\ln(1+\tan x) &= \ln\left[1 + \left(x + \frac{1}{3}x^3\right)\right] + x^3\varepsilon(x)\\ &= x + \frac{x^3}{3} - \frac{1}{2}\left(x + \frac{x^3}{3}\right)^2 + \frac{1}{3}\left(x + \frac{x^3}{3}\right)^3 + x^3\varepsilon(x).\end{aligned}$$

Then, $\quad \ln(1+\tan x) = x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + x^3\varepsilon(x).$

2° (a) We have

$$\left(\arctan\frac{1+x}{1-x}\right)' = \frac{1}{1+x^2}.$$

Hence, in the neighborhood of $x=0$ it may be written

$$\left(\arctan\frac{1+x}{1-x}\right)' = 1 - x^2 + x^2\varepsilon(x).$$

By integration, it yields that

$$\arctan\frac{1+x}{1-x} = \frac{\pi}{4} + x - \frac{x^3}{3} + x^3\varepsilon(x).$$

(b) We have

$$\cos\left(2\arctan\frac{1-x}{1+x}\right) = \cos\left(\frac{\pi}{2} + 2x - 2\frac{x^3}{3}\right) + x^3\varepsilon(x),$$

then

$$\begin{aligned}\cos\left(2\arctan\frac{1-x}{1+x}\right) &= -\sin\left(2x - \frac{2}{3}x^3\right) + x^3\varepsilon(x)\\ &= -2x + \frac{2}{3}x^3 + \frac{4}{3}x^3 + x^3\varepsilon(x).\end{aligned}$$

Hence, $\cos\left(2\arctan\dfrac{1-x}{1+x}\right) = -2x + 2x^3 + x^3\varepsilon(x)$.

3° Due to 1° and 2°, we have

$$\lim_{x\to 0}\frac{\ln(1+\tan x) + x + \dfrac{x^2}{2} + \cos\left(2\arctan\dfrac{1+x}{1-x}\right)}{x^3} = \frac{8}{3}.$$

Exercise 3

Put $t = \dfrac{1}{x}$ ($x \to +\infty \Leftrightarrow t \to 0_+$). This gives

$$f\left(\frac{1}{t}\right) = e^{-\frac{t}{1+2t}}\sqrt{\frac{1+2t}{t^2}} \quad \text{hence} \quad tf\left(\frac{1}{t}\right) = e^{-\frac{t}{1+2t}}\sqrt{1+2t}.$$

In neighborhood of $t=0$ and to order 2, we have

$$tf\left(\frac{1}{t}\right) = e^{-t+2t^2} \times \left(1 + t - \frac{1}{2}t^2\right) + t^2\varepsilon(t),$$

and thus,

$$tf\left(\frac{1}{t}\right) = \left(1 - t + \frac{5}{2}t^2\right)\left(1 + t - \frac{1}{2}t^2\right) + t^2\varepsilon(t) = 1 + t^2 + t^2\varepsilon(t).$$

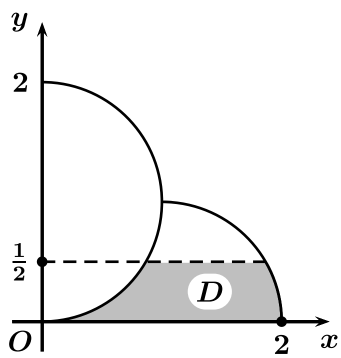
From which

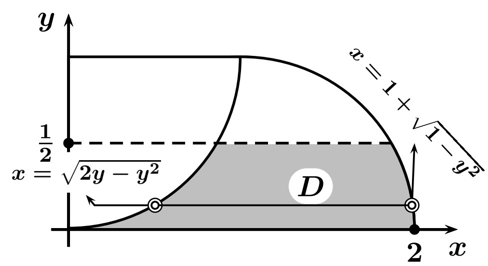
$$f(x) = x + \frac{1}{x} + \frac{1}{x}\varepsilon(x).$$

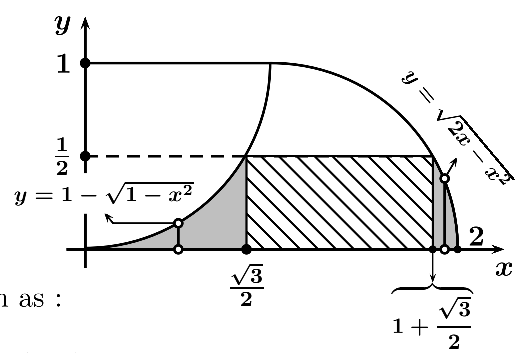
So the equation of the asymptote (Δ) is $y = x$.
Since $f(x) - y \simeq \dfrac{1}{x} > 0$ at $+\infty$, hence the curve (C) there is above its asymptote (Δ).

Exercise 4

1° The graphical representation of D :







2° The integral I is rewritten as :

$$I = \int_0^{\frac{1}{2}} \mathrm{d}y \int_{\sqrt{2y-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x;$$

or

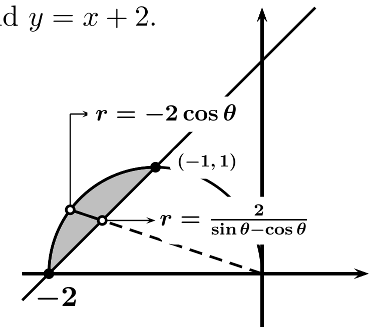
$$I = \int_0^{\frac{\sqrt{3}}{2}} \mathrm{d}x \int_0^{1-\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y + \int_{\frac{\sqrt{3}}{2}}^{1+\frac{\sqrt{3}}{2}} \mathrm{d}x \int_0^{\frac{1}{2}} f(x,y)\,\mathrm{d}y$$
$$+ \int_{1+\frac{\sqrt{3}}{2}}^{2} \mathrm{d}x \int_0^{\sqrt{2x-x^2}} f(x,y)\,\mathrm{d}y.$$

Exercise 5

1° The domain of integration D is bounded by the curves of equations : $x = -2$, $x = -1$, $y = \sqrt{-2x - x^2}$ and $y = x + 2$.

Hence D is defined by :

$$D : \begin{cases} \dfrac{3\pi}{4} \le \theta \le \pi; \\ \dfrac{2}{\sin\theta - \cos\theta} \le r \le -2\cos\theta. \end{cases}$$



2° $$I = \int_{\frac{3\pi}{4}}^{\pi} \mathrm{d}\theta \int_{\frac{2}{\sin\theta-\cos\theta}}^{-2\cos\theta} (\sin\theta + \cos\theta)\, \mathrm{d}r$$

$$= \int_{\frac{3\pi}{4}}^{\pi} (\sin\theta + \cos\theta)\left(-2\cos\theta - \frac{2}{\sin\theta - \cos\theta}\right) \mathrm{d}\theta.$$

Where,

- ▶ $-2\displaystyle\int_{\frac{3\pi}{4}}^{\pi} \sin\theta\cos\theta\,\mathrm{d}\theta = -\left[\sin^2\theta\right]_{\frac{3\pi}{4}}^{\pi} = \frac{1}{2}$
- ▶ $-2\displaystyle\int_{\frac{3\pi}{4}}^{\pi} \cos^2\theta\,\mathrm{d}\theta = -\left[\frac{\sin(2\theta)}{2} + \theta\right]_{\frac{3\pi}{4}}^{\pi} = -\frac{1}{4}\pi - \frac{1}{2}$
- ▶ $-2\displaystyle\int_{\frac{3\pi}{4}}^{\pi} \frac{\cos\theta + \sin\theta}{\sin\theta - \cos\theta}\,\mathrm{d}\theta = -2\left[\ln|\sin\theta - \cos\theta|\right]_{\frac{3\pi}{4}}^{\pi} = \ln 2$

From which, $I = \ln 2 - \dfrac{1}{4}\pi$.

Exercise 6

1° The considered differential equation is written as

$$y' = \frac{y}{x} + \tan\frac{y}{x}$$

which is a homogeneous diff. equation. Let $z = \dfrac{y}{x}$, and this yields $y' = xz' + z$. It results that

$$xz' = \tan z \quad \text{that is} \quad \frac{\mathrm{d}z}{\tan z} = \frac{\mathrm{d}x}{x}.$$

By integration, we find $\ln\left|\sin\dfrac{y}{x}\right| = \ln|x| + c$. If $y = \dfrac{\pi}{2}$ for $x = 1$, then $0 = 0 + c$, so $c = 0$.

2° Here it is a linear differential equation of the first order. Seeking the solution y in the form $y = u(x)v(x)$, we have

$$u'v + u\left[v' + \frac{1}{x(1+x)}v\right] = \frac{1+x}{x^3}.$$

We choose v such that $v' + \dfrac{1}{x(1+x)}v = 0$. This yields

$$\frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}x}{x(1+x)} = \left(\frac{1}{1+x} - \frac{1}{x}\right)\mathrm{d}x.$$

It follows that

$$\ln|v| = \ln\left|\frac{1+x}{x}\right| + \text{cnst}.$$

Then we may choose $v = \dfrac{1+x}{x}$.

Besides, $u'v = \dfrac{1+x}{x^3}$, so $u' = \dfrac{1}{x^2}$. From which $u = -\dfrac{1}{x} + c$, and consequently, $y = \left(-\dfrac{1}{x} + c\right)\dfrac{1+x}{x}$.

Finally, $y(1) = 1$ gives $c = \dfrac{3}{2}$.

LEBANESE UNIVERSITY Date: 09-09-2004

Faculty of Sciences (I)

Final II - session **BC11** Duration: 3 hours

ANSWER PAGE 35

Exercise 1 **[15 pts]**

1° Evaluate the following integrals :

$$I_1 = \int_1^2 \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x\,;$$

$$I_2 = \int_0^1 x^3\sqrt{1-x^2}\,\mathrm{d}x\,;$$

$$I_3 = \int_0^1 \arctan\sqrt{x}\,\mathrm{d}x.$$

2° (a) Calculate $J = \displaystyle\int_{\frac{1}{2}}^1 \frac{1}{t(1+t^2)}\,\mathrm{d}t$.

(b) Deduce the value of $K = \displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos^2 x}\,\mathrm{d}x$.

Exercise 2 **[20 pts]**

Let f be a function defined by :

$$f(x) = \frac{\ln\left(1+x+2x^2\right)+\cos(2x)+\sin^3 x - x - 1}{x^2}.$$

1° Give the finite expansion of order 2 of $f(x)$ in neighborhood of 0.

2° (a) Show that f could be extended by continuity to $x=0$. Let g be its extension function.

(b) Determine $g'(0)$.

3° Determine the equation of the tangent (T) to the representative curve C_g of g at the point of abscissa $x=0$, then precise the relative positions of C_g and (T) in neighborhood of $x=0$.

Exercise 3 ______________________________ **[15 pts]**

Determine the oblique asymptote (Δ) at $+\infty$ to the representative curve (C) of the function f, defined by :

$$f(x) = \sqrt{x^2 + 2x} - 3x^2 \left(\sqrt[3]{x^3 - x} - x \right),$$

and then precise the relative position of (C) and (Δ) at $+\infty$.

Exercise 4 ______________________________ **[20 pts]**

Let D be the domain in the plane $\mathbb{R}^2$, defined by :

$$D = \left\{ (x, y) \in \mathbb{R}^2 \ : x + y \geq 2, \quad x^2 + y^2 \leq 4 \ \text{ and } \ (x-1)^2 + y^2 \geq 1 \right\}.$$

1° Sketch D.

2° Using cartesian coordinates, write down $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ in two different forms.

3° Using polar coordinates, rewrite down $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

Exercise 5 ______________________________ **[15 pts]**

Let $I = \displaystyle\int_0^1 \mathrm{d}x \int_x^1 e^{\frac{x}{y}}\,\mathrm{d}y$.

1° Sketch the domain of integration D.

2° Calculate I.

Exercise 6 ______________________________ **[15 pts]**

1° Solve the differential equation

$$(x^2 + xy)y' - (x^2 + y^2) = 0.$$

Find the particular solution, that verifies $y(1) = 2$.

2° Solve the differential equation

$$(1 + x^2)^2\, y' + 2xy = xe^{\frac{1}{1+x^2}}.$$

Find the particular solution, that verifies $y(0) = e$.

Exercise 1

1° **Evaluation of I_1.** Integrating by parts, with

$$\begin{cases} u = \ln x \\ \mathrm{d}v = \dfrac{\mathrm{d}x}{\sqrt{x}} \end{cases} \quad \text{thus,} \quad \begin{cases} \mathrm{d}u = \dfrac{\mathrm{d}x}{x}; \\ v = 2\sqrt{x}. \end{cases}$$

The formula of integration by parts permits us writing,

$$\begin{aligned} I_1 &= \left[2\sqrt{x}\ln x\right]_1^2 - 2\int_1^2 \frac{\sqrt{x}}{x}\,\mathrm{d}x \\ &= 2\sqrt{2}\ln 2 - 2\int_1^2 \frac{\mathrm{d}x}{\sqrt{x}}. \end{aligned}$$

So $I_1 = 2\sqrt{2}\ln 2 - 4\left(\sqrt{2}-1\right)$.

Evaluation of I_2. We may write,

$$I_2 = \int_0^1 x^2\sqrt{1-x^2}\,x\,\mathrm{d}x.$$

Performing the change of variable $t = 1 - x^2$, we obtain

$$I_2 = -\frac{1}{2}\int_1^0 (1-t)\sqrt{t}\,\mathrm{d}t = \frac{2}{15}.$$

Evaluation of I_3. Integrating by parts, with

$$\begin{cases} u = \arctan\sqrt{x} \\ \mathrm{d}v = \mathrm{d}x \end{cases} \quad \text{thus,} \quad \begin{cases} \mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{x}(1+x)}; \\ v = x. \end{cases}$$

The formula of integration by parts permits us writing,

$$I_3 = \left[x\arctan\sqrt{x}\right]_0^1 - \frac{1}{2}\int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x.$$

Set $t = \sqrt{x}$. Thus $\mathrm{d}t = \dfrac{\mathrm{d}x}{2\sqrt{x}}$, and $\mathrm{d}x = 2t\,\mathrm{d}t$. Hence,

$$\int_0^1 \frac{\sqrt{x}}{1+x}\,\mathrm{d}x = 2\int_0^1 \frac{t^2}{1+t^2}\,\mathrm{d}t = 2\int_0^1 \frac{1+t^2-1}{1+t^2}\,\mathrm{d}t = 2 - \frac{\pi}{2}.$$

Finally, $I_3 = \dfrac{\pi}{2} - 1$.

2° (a) $J = \int_{\frac{1}{2}}^{1} \frac{1+t^2-t^2}{t(1+t^2)}\,\mathrm{d}t = \int_{\frac{1}{2}}^{1} \left[\frac{1}{t} - \frac{t}{1+t^2}\right] \mathrm{d}t$. From which,

$$J = \left[\ln|t| - \frac{1}{2}\ln\left|1+t^2\right|\right]_{\frac{1}{2}}^{1} = -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 5.$$

(b) Let $t = \cos x$ we find

$$\begin{aligned} K &= \int_0^{\frac{\pi}{3}} \frac{\sin x\,\mathrm{d}x}{\cos x(\cos^2 x + 1)} \\ &= -\int_1^{\frac{1}{2}} \frac{\mathrm{d}t}{t(1+t^2)} = J. \end{aligned}$$

Exercise 2

1° In the neighborhood of $x = 0$, we have

$$\begin{aligned} \ln(1+x+2x^2) &= x + 2x^2 - \frac{(x+2x^2)^2}{2} + \frac{(x+2x^2)^3}{3} - \frac{(x+2x^2)^4}{4} + x^4\varepsilon(x) \\ &= x + \frac{3}{2}x^2 - \frac{5}{3}x^3 - \frac{1}{4}x^4 + x^4\varepsilon(x); \end{aligned}$$

$$\begin{aligned} \cos(2x) &= 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + x^4\varepsilon(x) \\ &= 1 - 2x^2 + \frac{2}{3}x^4 + x^4\varepsilon(x); \end{aligned}$$

$$\sin^3 x = x^3 + x^4\varepsilon(x).$$

Hence,

$$f(x) = -\frac{1}{2} - \frac{2}{3}x + \frac{5}{12}x^2 + x^2\varepsilon(x).$$

2° (a) We have

$$\lim_{x\to 0} f(x) = -\frac{1}{2}.$$

Thus the function f could be extended by continuity to $x = 0$ by the function g, defined by :

$$g(x) = \begin{cases} f(x) & \text{if } \quad x \neq 0; \\ -\dfrac{1}{2} & \text{if } \quad x = 0. \end{cases}$$

(b) $g'(0) = -\frac{2}{3}$.

3° The equation of the tangent is :

$$y = -\frac{1}{2} - \frac{2}{3}x.$$

In neighborhood of $x = 0$, $f(x) - y \simeq \dfrac{5}{12}x^2 > 0$. The representative curve of g is thus above the tangent in neighborhood of $x = 0$.

Exercise 3

1° Set $t = \dfrac{1}{x}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. It yields

$$\begin{aligned} f\left(\frac{1}{t}\right) &= \sqrt{\frac{1+2t}{t^2}} - \frac{3}{t^2}\left(\sqrt[3]{\frac{1-t^2}{t^3}} - \frac{1}{t}\right) \\ &= \frac{1}{t}\sqrt{1+2t} - \frac{3}{t^3}\left(\sqrt[3]{1-t^2} - 1\right). \end{aligned}$$

Thus,

$$tf\left(\frac{1}{t}\right) = \sqrt{1+2t} - \frac{3}{t^2}\left(\sqrt[3]{1-t^2} - 1\right).$$

In neighborhood of $t = 0$, we have

$$\begin{aligned} \sqrt{1+2t} &= 1 + t - \frac{1}{2}t^2 + t^2\varepsilon(t) \\ \sqrt[3]{1-t^2} &= 1 - \frac{1}{3}t^2 - \frac{1}{9}t^4 + t^4\varepsilon(t). \end{aligned}$$

Hence,

$$\begin{aligned} tf\left(\frac{1}{t}\right) &= 1 + t - \frac{1}{2}t^2 + 1 + \frac{1}{3}t^2 + t^2\varepsilon(t) \\ &= 2 + t - \frac{1}{6}t^2 + t^2\varepsilon(t). \end{aligned}$$

From which,

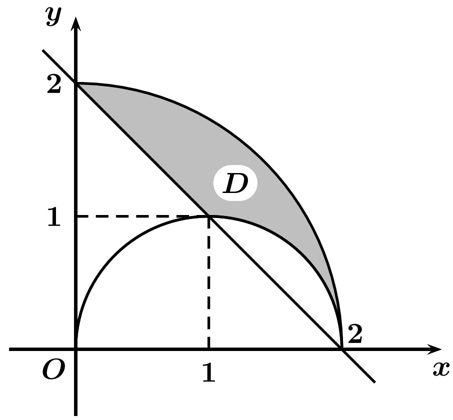
$$f(x) = 2x + 1 - \frac{1}{6x} + \frac{1}{x}\varepsilon(x).$$

So the equation of the oblique asymptote (Δ) is $y = 2x + 1$.

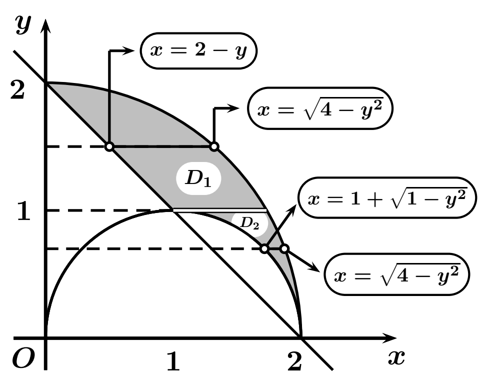
2° At $+\infty$, $f(x) - y \simeq -\dfrac{1}{6x} < 0$, then the curve (C) is below its asymptote (Δ).

Exercise 4

1° Graphical representation of D :



2° We have



$$D = D_1 \cup D_2$$

where D_1 and D_2 are defined by:

$$D_1 : \begin{cases} 1 \leq y \leq 2; \\ 2 - y \leq x \leq \sqrt{4 - y^2}, \end{cases}$$

and

$$D_2 : \begin{cases} 0 \leq y \leq 1; \\ 1 + \sqrt{1 - y^2} \leq x \leq \sqrt{4 - y^2}. \end{cases}$$

The integral I may be written as

$$I = \int_0^1 \mathrm{d}y \int_{1+\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x, y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{2-y}^{\sqrt{4-y^2}} f(x, y)\,\mathrm{d}x.$$

Besides, we have

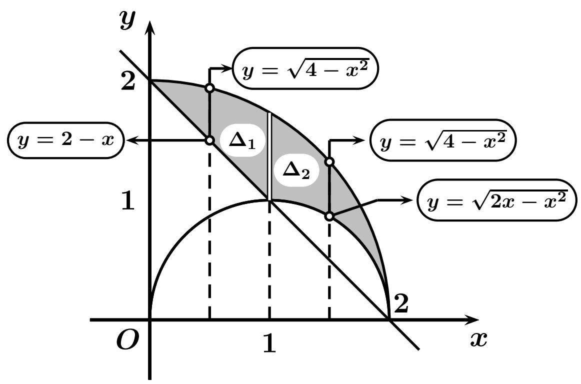
$$D = \Delta_1 \cup \Delta_2,$$

where Δ_1 and Δ_2 are defined by :

$$\Delta_1 : \begin{cases} 0 \leq x \leq 1 \\ 2 - x \leq y \leq \sqrt{4 - x^2}\,, \end{cases}$$

and

$$\Delta_2 : \begin{cases} 1 \le x \le 2 \\ \sqrt{2x - x^2} \le y \le \sqrt{4 - x^2}\,. \end{cases}$$



So the integral I is written as

$$I = \int_0^1 \mathrm{d}x \int_{2-x}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y + \int_1^2 \mathrm{d}x \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y.$$

3° Using polar coordinates, we have

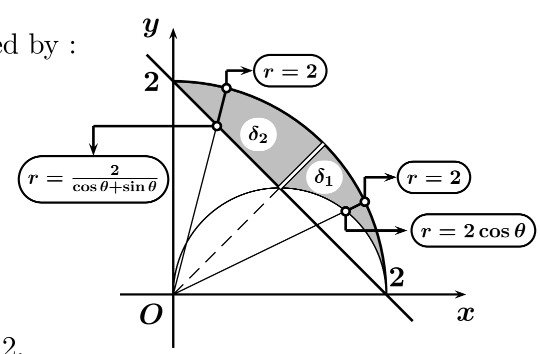
$$D = \delta_1 \cup \delta_2$$

where δ_1 and δ_2 are defined by :

$$\delta_1 : \begin{cases} 0 \le \theta \le \dfrac{\pi}{4}; \\ 2\cos\theta \le r \le 2, \end{cases}$$

and

$$\delta_2 : \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}; \\ \dfrac{2}{\cos\theta + \sin\theta} \le r \le 2. \end{cases}$$

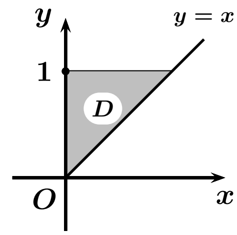


The integral I may be written as

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{2\cos\theta}^{2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r \\ &\quad + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{2}{\cos\theta+\sin\theta}}^{2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r \end{aligned}$$

Exercise 5

1° The domain of integration D is bounded by the curves of equations : $x = 0$, $x = 1$, $y = x$ and $y = 1$.



2° $$I = \int_0^1 \mathrm{d}y \int_0^y e^{\frac{x}{y}}\,\mathrm{d}x = \int_0^1 \left[y e^{\frac{x}{y}} \right]_{x=0}^{x=y} \mathrm{d}y$$
$$= (e-1)\int_0^1 y\,\mathrm{d}y = \frac{e-1}{2}.$$

Exercise 6

1° Rewriting the considered differential equation as

$$y' = \frac{1 + \frac{y^2}{x^2}}{1 + \frac{y}{x}}$$

which is obviously a homogeneous differential equation. Let $z = \dfrac{y}{x}$, so we get $y' = xz' + z$. Thus,

$$xz' + z = \frac{1+z^2}{1+z} \quad \text{hence} \quad \frac{z+1}{z-1}\,\mathrm{d}z = -\frac{\mathrm{d}x}{x}.$$

By integration, it is found $\dfrac{y}{x} + 2\ln\left|1 - \dfrac{y}{x}\right| = -\ln|x| + c$. If, $y = 2$ for $x = 1$, then $c = 2$.

2° It is the case of a first order linear differential equation. Seeking the solution y in the form $y = u(x)v(x)$. It yields

$$u'v + u\left[v' + \frac{2x}{(1+x^2)^2}v\right] = \frac{xe^{\frac{1}{1+x^2}}}{(1+x^2)^2}.$$

We choose v such that $v' + \dfrac{2x}{(1+x^2)^2}v = 0$. From which

$$\frac{\mathrm{d}v}{v} = -\frac{2x\,\mathrm{d}x}{(1+x^2)^2} \quad \text{so} \quad v = e^{\frac{1}{1+x^2}}.$$

Moreover, $u'v = \dfrac{xe^{\frac{1}{1+x^2}}}{(1+x^2)^2}$, implies that $u' = \dfrac{x}{(1+x^2)^2}$.

Hence $u = \dfrac{-1}{2(1+x^2)} + c$, and consequently $y = \left(\dfrac{-1}{2(1+x^2)} + c\right) e^{\frac{1}{1+x^2}}$.

Finally, $y(0) = e$ gives $c = \dfrac{3}{2}$.

LEBANESE UNIVERSITY Date: 30-06-2005
Faculty of Sciences (I)

Final I-session **BC11** **Duration: 3 hours**

ANSWER PAGE 43

Exercise 1 **[15 pts]**

Calculate the following integrals :

$$I_1 = \int x \ln \frac{1+x}{1-x}\,\mathrm{d}x; \quad I_2 = \int_0^{\frac{\pi}{4}} \frac{1+\tan x}{5+4\cos(2x)}\,\mathrm{d}x; \quad I_3 = \int_1^{+\infty} \frac{\arctan x}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

Exercise 2 **[15 pts]**

Let a function f be defined by

$$f(x) = \begin{cases} e^{\frac{\sqrt[3]{1-3x}-1}{x}} + \cos(x+x^2) - \dfrac{1}{x}\ln\dfrac{e^x-1}{x} & \text{if} \quad x \neq 0; \\ \dfrac{1}{2} + \dfrac{1}{e} & \text{if} \quad x = 0. \end{cases}$$

1° Give the finite expansion of order 2 of f in neighborhood of 0.

2° Show that f is continuous at 0.

3° Show that f is differentiable at 0. Determine $f'(0)$.

4° Determine the equation of the tangent (Δ) to the representative curve (C) of f, at the point of abscissa $x = 0$, and precise the relative position of (Δ) and (C) in neighborhood of 0.

Exercise 3 **[10 pts]**

Let a function f be defined by :

$$f(x) = x\left(e^{\frac{1}{x}} - 1\right)\left(\sqrt{1+x^2} - x\right).$$

Use finite expansions to determine the asymptotes to the representative curve (C) of the function f, in the neighborhoods of $\pm\infty$. Precise the relative positions of these asymptotes with respect to (C).

Exercise 4 **[12 pts]**

1° Solve the following differential equations :

(a) $(1 + e^{-y})\,\mathrm{d}x - (x-1)\,\mathrm{d}y = 0$.

(b) $xyy' + x^2 + y^2 = 0$.

2° Integrate the following differential equation :

$$y' - \frac{1}{x(1+x)}y = \frac{\ln x}{1+x}.$$

Determine the particular solution that verifies $y(1) = 1$.

Exercise 5 **[6 pts]**

Calculate $\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{d}x \int_x^{\frac{\pi}{2}} \cos^{99} y \cos x \,\mathrm{d}y$.

Exercise 6 **[17 pts]**

Let

$$I = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{1-\sqrt{1-y^2}} \mathrm{d}x.$$

1° Sketch the domain of integration D.

2° Rewrite the expression of I by changing the order of integration.

3° Using polar coordinates, calculate I.

Solution

Exercise 1

Evaluation of I_1. To integrate by parts we set

$$\begin{cases} u = \ln \dfrac{1+x}{1-x} \\ \mathrm{d}v = x\,\mathrm{d}x \end{cases} \quad \text{thus,} \quad \begin{cases} \mathrm{d}u = \dfrac{2\,\mathrm{d}x}{1-x^2} \\ v = \dfrac{x^2}{2}. \end{cases}$$

The formula of integration by parts gives

$$I_1 = \frac{x^2}{2}\ln\frac{1+x}{1-x} - \int \frac{x^2}{1-x^2}\,\mathrm{d}x$$

that is,

$$I_1 = x + \frac{x^2}{2}\ln\frac{1+x}{1-x} - \frac{1}{2}\ln\frac{1+x}{1-x} + \text{cnst}.$$

Evaluation of I_2. Let $t = \tan x$, thus $\mathrm{d}t = (1+t^2)\,\mathrm{d}x$. And so,

$$I_2 = \int_0^1 \frac{1+t}{5+4\dfrac{1-t^2}{1+t^2}}\frac{\mathrm{d}t}{1+t^2} = \int_0^1 \frac{1+t}{9+t^2}\,\mathrm{d}t$$

then

$$I_2 = \int_0^1 \frac{\mathrm{d}t}{9+t^2} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}(9+t^2)}{9+t^2} = \left[\frac{1}{3}\arctan\frac{t}{3}\right]_0^1 + \frac{1}{2}\left[\ln(9+t^2)\right]_0^1.$$

Finally, $I_2 = \dfrac{1}{3}\arctan\dfrac{1}{3} + \dfrac{1}{2}\ln\dfrac{10}{9}$.

Evaluation of I_3. Setting $t = \arctan x$, we obtain

$$I_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t\cos t\,\mathrm{d}t \underset{\text{by parts}}{=} \Big[t\sin t\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin t\,\mathrm{d}t.$$

Thus, $\quad I_3 = \dfrac{\pi}{2} - \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{4}+1\right).$

Exercise 2

1° $\cos(x+x^2) = 1 - \dfrac{x^2}{2} + x^2\varepsilon(x).$

$$e^{\frac{\sqrt[3]{1-3x}-1}{x}} = e^{-1-x-\frac{5}{3}x^2} + x^2\varepsilon(x) = \frac{1}{e}\left(1 - x - \frac{7x^2}{6}\right) + x^2\varepsilon(x).$$

$$\frac{1}{x}\ln\frac{e^x-1}{x} = \frac{1}{x}\ln\left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24}\right) + x^2\varepsilon(x) = \frac{1}{2} + \frac{x}{24} + x^2\varepsilon(x).$$

It gives $f(x) = \dfrac{1}{2} + \dfrac{1}{e} - \left(\dfrac{1}{e} + \dfrac{1}{24}\right)x - \left(\dfrac{7}{6e} + \dfrac{1}{2}\right)x^2 + x^2\varepsilon(x).$

2° f is continuous at $x = 0$, since $\lim\limits_{x\to 0} f(x) = \dfrac{1}{2} + \dfrac{1}{e} = f(0).$

3° $f'(0) = -\dfrac{1}{e} - \dfrac{1}{24}.$

4° The equation of the tangent (Δ) to the representative curve of f, at the point of abscissa $x = 0$ is given by, $y = \dfrac{1}{2} + \dfrac{1}{e} - \left(\dfrac{1}{e} + \dfrac{1}{24}\right)x$, and the term $-\left(\dfrac{7}{6e} + \dfrac{1}{2}\right)x^2 < 0$ shows that the curve is below that tangent in the neighborhood of 0.

Exercise 3

Let $t = \dfrac{1}{x}$. We look for the f.e. of order 2 of $tf\left(\dfrac{1}{t}\right)$ as $x \to \infty$.

$$tf\left(\frac{1}{t}\right) = (e^t - 1)\left(\frac{\sqrt{1+t^2}}{|t|} - \frac{1}{t}\right).$$

As $x \to +\infty$

$$tf\left(\frac{1}{t}\right) = \frac{t^2}{2} + t^2\varepsilon(t), \quad \text{thus} \quad f(x) = \frac{1}{2x} + \frac{1}{x}\varepsilon(x), \quad \text{with} \quad \lim_{x\to+\infty}\varepsilon(x) = 0.$$

The curve admits the horizontal asymptote $y = 0$, and $\dfrac{1}{2x} > 0$ implies that the curve is above this asymptote at $+\infty$.

As $x \to -\infty$

$$tf\left(\frac{1}{t}\right) = -\frac{1}{t}\left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)\left(2 + \frac{t^2}{2}\right) + t^2\varepsilon(t) = -2 - t - \frac{5}{6}t^2 + t^2\varepsilon(t),$$

thus $f(x) = -1 - 2x - \dfrac{5}{6x} + \dfrac{1}{x}\varepsilon(x) \quad$ with $\lim\limits_{x\to-\infty}\varepsilon(x) = 0.$

The curve admits the oblique asymptote $y = -1 - 2x$, and since $-\dfrac{5}{6x} > 0$ then the curve is above its asymptote at $-\infty$.

Exercise 4

1° (a) Separating the variables, we get

$$\frac{e^y\,\mathrm{d}y}{1+e^y} = \frac{\mathrm{d}x}{x-1}$$

thus, $y = \ln|k(1-x)-1|$, where k is an arbitrary constant.

(b) The considered differential equation is homogeneous. In fact,

$$y' = -\frac{x}{y} - \frac{y}{x}.$$

Set $y = zx$. It gives,

$$\frac{z\,\mathrm{d}z}{1+2z^2} = -\frac{\mathrm{d}x}{x}.$$

By integration we get, $\quad \dfrac{1}{4}\ln\left(1+2\dfrac{y^2}{x^2}\right) = -\ln|x| + c.$

2° The considered differential equation is linear of first order. Seeking the solution y in the form $y = u(x)v(x)$ where

- $v' - \frac{1}{x(1+x)}v = 0$, then $\ln v = \displaystyle\int \frac{\mathrm{d}x}{x(1+x)} = \ln\frac{x}{1+x}$ and so, $v = \dfrac{x}{1+x}$.
- $u'v = \dfrac{\ln x}{1+x}$, then $u' = \dfrac{\ln x}{x}$, and so $u = \displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{(\ln x)^2}{2} + c.$

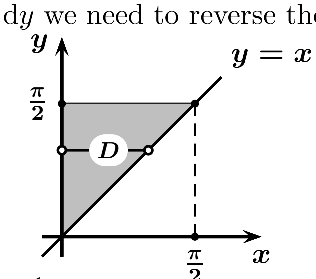
Finally : $y = x\dfrac{(\ln x)^2 + c}{2(1+x)}$.

$y(1) = 1$ gives $c = 4$.

Exercise 5

The integration domain D is bounded by the curves whose equations are : $x = 0$, $x = \dfrac{\pi}{2}$, $y = x$ and $y = \dfrac{\pi}{2}$.

To be able to evaluate $I = \displaystyle\int_0^{\frac{\pi}{2}} \mathrm{d}x \int_x^{\frac{\pi}{2}} \cos^{99} y \cos x\,\mathrm{d}y$ we need to reverse the order of integration. Therefore,

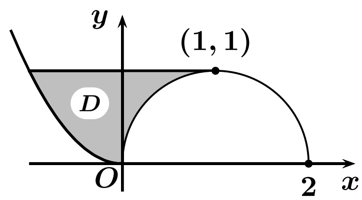


$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \cos^{99} y\,\mathrm{d}y \int_0^y \cos x\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos^{99} y\,\mathrm{d}y \Big[\sin x\Big]_0^y \\ &= \int_0^{\frac{\pi}{2}} \cos^{99} y \sin y\,\mathrm{d}y = -\left[\frac{\cos^{100} y}{100}\right]_0^{\frac{\pi}{2}} = \frac{1}{100}. \end{aligned}$$

Exercise 6

1° The domain of integration D of the integral I is bounded by the curves of equations :

$$\begin{cases} y = 0, \quad y = 1; \\ x = -\sqrt{y} \quad \text{and} \quad x = 1 - \sqrt{1-y^2}. \end{cases}$$



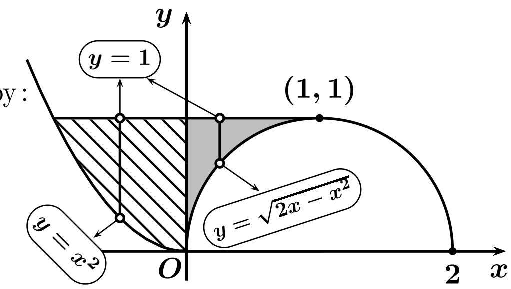
2° By a suitable partitioning of (D), we have

$$D = D_1 \cup D_2$$

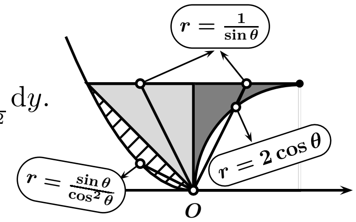
where D_1 and D_2 are defined by:

$$D_1 : \begin{cases} 0 \leq x \leq 1 \\ \sqrt{2x - x^2} \leq y \leq 1 \end{cases}$$

$$D_2 : \begin{cases} -1 \leq x \leq 0 \\ x^2 \leq y \leq 1 \end{cases}$$



Thus, $I = \displaystyle\int_{-1}^{0} \mathrm{d}x \int_{x^2}^{1} \mathrm{d}y + \int_{0}^{1} \mathrm{d}x \int_{\sqrt{2x-x^2}}^{1} \mathrm{d}y.$



3° In polar coordinates

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{2\cos\theta}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r + \int_{\frac{\pi}{2}}^{3\frac{\pi}{4}} \mathrm{d}\theta \int_{0}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r + \int_{3\frac{\pi}{4}}^{\pi} \mathrm{d}\theta \int_{0}^{\frac{\sin\theta}{\cos^2\theta}} r\,\mathrm{d}r$$

with

$$\triangleright\ I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{2\cos\theta}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2\theta} - 4\cos^2\theta\right) \mathrm{d}\theta$$

$$= \frac{1}{2}\Big[-\cot\theta - \sin(2\theta) - 2\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 1 - \frac{\pi}{4}.$$

$$\triangleright\ I_2 = \int_{\frac{\pi}{2}}^{3\frac{\pi}{4}} \mathrm{d}\theta \int_{0}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{2}}^{3\frac{\pi}{4}} \frac{1}{\sin^2\theta}\,\mathrm{d}\theta = -\frac{1}{2}\big[\cot\theta\big]_{\frac{\pi}{2}}^{3\frac{\pi}{4}} = \frac{1}{2}.$$

$$\triangleright\ I_3 = \int_{3\frac{\pi}{4}}^{\pi} \mathrm{d}\theta \int_{0}^{\frac{\sin\theta}{\cos^2\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{2}}^{3\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^4\theta}\,\mathrm{d}\theta = \frac{1}{6}\Big[\tan^3\theta\Big]_{3\frac{\pi}{4}}^{\pi} = \frac{1}{6}.$$

Therefore, $I = \dfrac{5}{3} - \dfrac{\pi}{4}$.

LEBANESE UNIVERSITY Date: 14-09-2005

Faculty of Sciences (I)

Final II - session **BC11** **Duration: 3 hours**

ANSWER PAGE 49

Exercise 1 **[18 pts]**

Calculate the following integrals :

$$I_1 = \int_{-1}^{0} x\sqrt{1+x}\,\mathrm{d}x; \quad I_2 = \int_{0}^{\frac{\pi}{2}} \frac{\sin(2x)}{\sqrt{1+3\cos^2 x}}\,\mathrm{d}x; \quad I_3 = \int_{0}^{\frac{\pi}{4}} \tan x \ln(\cos x)\,\mathrm{d}x.$$

Exercise 2 **[17 pts]**

Give a function f defined by :

$$f(x) = \arctan(1+x) - \frac{1}{2}\cos(x\sqrt{2}) - \frac{x}{2}e^{\frac{x}{2}}.$$

1° Give the f.e. of order 3 of f in the neighborhood of 0.

2° Determine the equation of the tangent (Δ) to the representative curve (C) of f, at the point of abscissa $x = 0$, and precise the relative position of (Δ) and (C) in neighborhood of 0.

Exercise 3 **[14 pts]**

Use finite expansions to calculate the following limits :

1° $\displaystyle\lim_{x\to 0} \frac{1}{x}\left[\frac{1}{(1+x)e^x} - \frac{1}{\cos x}\right]$.

2° $\displaystyle\lim_{x\to +\infty} x^2\left[e^{\frac{1}{x}} - e^{\frac{1}{x-1}}\right]$.

Exercise 4 **[8 pts]**

Consider

$$D = \left\{(x,y) \in \mathbb{R}^2 \ ; \ 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\right\}.$$

Calculate $\displaystyle I = \iint_D x\cos(xy)\cos^2 x\,\mathrm{d}x\,\mathrm{d}y.$

Exercise 5 ______________________________ **[25 pts]**

Let

$$I = \int_0^1 \mathrm{d}y \int_{-\sqrt{2y-y^2}}^{1-\sqrt{1-y^2}} \mathrm{d}x.$$

1° Sketch the domain of integration D.

2° Rewrite the expression of I, by changing the order of integration.

3° Using polar coordinates, calculate I.

Exercise 6 ______________________________ **[18 pts]**

1° Solve the following differential equations :

(a) $(x+2y)y' + y + 2x = 0$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{\dfrac{1-y}{1+x^2}}$.

2° Integrate the following differential equation :

$$y' - \frac{1+x}{x}y = x^2.$$

Determine the particular solution that verifies $y(-1) = 1$.

Solution

Exercise 1

Evaluation of I_1. Set $t=\sqrt{1+x}$, hence

$$x=t^2-1, \quad \text{and} \quad \mathrm{d}x=2t\,\mathrm{d}t.$$

That yields : $I_1=2\int_0^1 (t^2-1)t^2\,\mathrm{d}t=2\left[\frac{t^5}{5}-\frac{t^3}{3}\right]_0^1=2\left(\frac{1}{5}-\frac{1}{3}\right)=-\frac{4}{15}$.

Evaluation of I_2. Let $t=1+3\cos^2 x$, so $\mathrm{d}t=-3\sin(2x)\,\mathrm{d}x$.

Hence : $I_2=-\frac{1}{3}\int_4^1 \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}t=-\frac{1}{3}\left[2\sqrt{t}\right]_4^1=-\frac{1}{3}(2-4)=\frac{2}{3}$.

Another method : $I_2 \underset{t=\cos x}{=} -2\int_1^0 \frac{t\,\mathrm{d}t}{\sqrt{1+3t^2}}=\left[-\frac{2}{3}\sqrt{1+3t^2}\right]_1^0=\frac{2}{3}$.

Evaluation of I_3.

First method : Performing the change of variable

$$t=\ln(\cos x) \quad \text{then,} \quad \mathrm{d}t=-\tan x\,\mathrm{d}x.$$

Hence : $I_3=-\int_0^{-\frac{1}{2}\ln 2} t\,\mathrm{d}t=-\left[\frac{t^2}{2}\right]_0^{-\frac{1}{2}\ln 2}=-\frac{1}{8}\ln^2 2$.

Second method : Integrating by parts, with

$$\begin{cases} u &= \ln(\cos x) \\ \mathrm{d}v &= \tan x\,\mathrm{d}x \end{cases} \quad \text{which gives} \quad \begin{cases} \mathrm{d}u &= -\dfrac{\sin x}{\cos x}\,\mathrm{d}x; \\ v &= -\ln(\cos x). \end{cases}$$

Then we get by the formula of integration by parts

$$\begin{aligned} I_3 &= \left[uv\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} v\,\mathrm{d}u \\ &= -\left[\ln^2(\cos x)\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}} \tan x\ln(\cos x)\,\mathrm{d}x=-\frac{1}{4}\ln^2 2-I_3. \end{aligned}$$

Finally, $\quad I_3=-\frac{1}{8}\ln^2 2$.

Third method : With $t=\cos x$,

$$I_3=-\int_1^{\frac{\sqrt{2}}{2}} \ln t\frac{\mathrm{d}t}{t}=-\left.\frac{(\ln t)^2}{2}\right]_1^{\frac{\sqrt{2}}{2}}=-\frac{1}{8}\ln^2 2.$$

Exercise 2

1° Having $\arctan'(1+x) = \dfrac{1}{2+2x+x^2} = \dfrac{1}{2} - \dfrac{x}{2} + \dfrac{x^2}{4} + x^2\varepsilon(x)$, we obtain $\arctan(1+x) = \dfrac{\pi}{4} + \dfrac{x}{2} - \dfrac{x^2}{4} + \dfrac{x^3}{12} + x^3\varepsilon(x)$.

Similarly, in neighborhood of zero, and up to order 3, we have

$$\cos(\sqrt{2}x) = 1 - x^2 + x^3\varepsilon(x) \quad \text{and} \quad xe^{\frac{x}{2}} = x + \frac{x^2}{2} + \frac{x^3}{8} + x^3\varepsilon(x).$$

Then :

$$f(x) = \frac{\pi}{4} - \frac{1}{2} + \frac{x^3}{48} + x^3\varepsilon(x) \quad \text{with} \quad \lim_{x\to 0}\varepsilon(x) = 0.$$

2° The equation of the tangent (Δ) to the representative curve of f, at the point of abscissa $x = 0$, is

$$y = \frac{\pi}{4} - \frac{1}{2}.$$

As $x \to 0$, $f(x) - y \simeq \dfrac{x^3}{48}$, then to the left of 0, the curve is below (Δ), and to the right of 0, the curve is above (Δ).

Exercise 3

1° We may write $\dfrac{1}{x}\left[\dfrac{1}{(1+x)e^x} - \dfrac{1}{\cos x}\right] = \dfrac{\cos x - (1+x)e^x}{x(1+x)e^x\cos x}$.

In neighborhood of $x = 0$, and to order 1, we have

$$\begin{aligned}\cos x - (1+x)e^x &= 1 - (1+x)(1+x) + x\varepsilon(x) = -2x + x\varepsilon(x).\\ x(1+x)e^x\cos x &= x + x\varepsilon(x).\end{aligned}$$

From which, $\displaystyle\lim_{x\to 0}\frac{1}{x}\left[\frac{1}{(1+x)e^x} - \frac{1}{\cos x}\right] = \lim_{x\to 0}\left(-2 + \varepsilon(x)\right) = -2.$

2° Setting $t = \dfrac{1}{x}$, then $x^2\left(e^{\frac{1}{x}} - e^{\frac{1}{x-1}}\right) = \dfrac{e^t - e^{\frac{t}{1-t}}}{t^2}$.

In neighborhood of $t = 0$, and up to order 2, we have :

$$e^t = 1 + t + \frac{t^2}{2} + t^2\varepsilon(t).$$

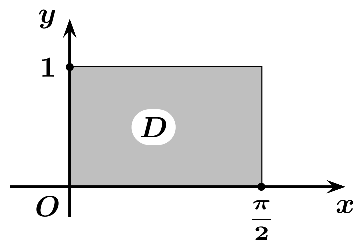
$$e^{\frac{t}{1-t}} = e^{t+t^2} + t^2\varepsilon(t) = 1 + t + \frac{3t^2}{2} + t^2\varepsilon(t).$$

From which we deduce that $\displaystyle\lim_{x\to+\infty} x^2\left(e^{\frac{1}{x}} - e^{\frac{1}{x-1}}\right) = -1.$

Exercise 4

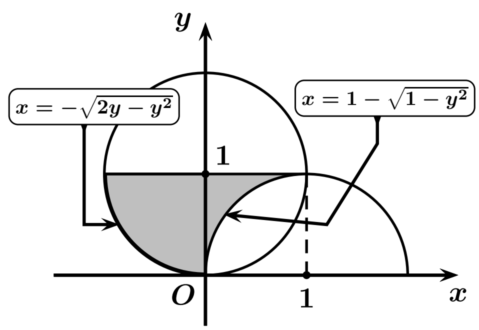
The domain of integration D of the integral I is bounded by the lines of equations : $x = 0$, $x = \dfrac{\pi}{2}$, $y = 0$ and $y = 1$.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} x\cos^2(x)\,\mathrm{d}x \int_0^1 \cos(xy)\,\mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} \cos^2 x\big[\sin(xy)\big]_{y=0}^{y=1}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \cos^2 x \sin x\,\mathrm{d}x \\ &= \left[-\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{1}{3}. \end{aligned}$$



Exercise 5

1° The domain of integration D, of the integral I, is bounded by the curves of equations : $y = 0$, $y = 1$, $x = -\sqrt{2y - y^2}$ and $x = 1 - \sqrt{1 - y^2}$.



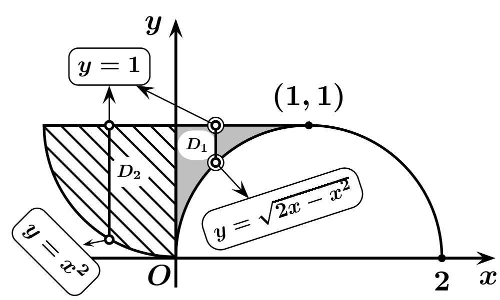
2° We may write

$$D = D_1 \cup D_2,$$

where D_1 and D_2 are defined by :

$$D_1 : \begin{cases} 0 \le x \le 1 \\ \sqrt{2x - x^2} \le y \le 1 \end{cases}$$

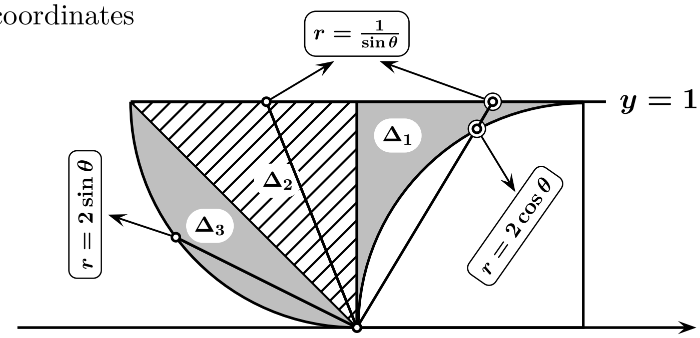
$$D_2 : \begin{cases} -1 \le x \le 0 \\ 1 - \sqrt{1 - x^2} \le y \le 1 \end{cases}$$



From which

$$I = \int_{-1}^{0} \mathrm{d}x \int_{1-\sqrt{1-x^2}}^{1} \mathrm{d}y + \int_0^1 \mathrm{d}x \int_{\sqrt{2x-x^2}}^{1} \mathrm{d}y.$$

3° In polar coordinates



We may write

$$D = \Delta_1 \cup \Delta_2 \cup \Delta_3\,,$$

where Δ_1, Δ_2 and Δ_3 are defined by :

$$\Delta_1 : \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2} \\ 2\cos\theta \le r \le \dfrac{1}{\sin\theta} \end{cases} \qquad \Delta_2 : \begin{cases} \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{4} \\ 0 \le r \le \dfrac{1}{\sin\theta} \end{cases}$$

and $\quad \Delta_3 : \begin{cases} \dfrac{3\pi}{4} \le \theta \le \pi \\ 0 \le r \le 2\sin\theta \end{cases}$

It follows that,

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{2\cos\theta}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \mathrm{d}\theta \int_0^{\frac{1}{\sin\theta}} r\,\mathrm{d}r + \int_{\frac{3\pi}{4}}^{\pi} \mathrm{d}\theta \int_0^{2\sin\theta} r\,\mathrm{d}r.$$

On the other hand,

$$\triangleright \quad I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{2\cos\theta}^{\frac{1}{\sin\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2\theta} - 4\cos^2\theta\right) \mathrm{d}\theta$$

$$= \frac{1}{2}\Big[-\cot\theta - \sin(2\theta) - 2\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 1 - \frac{\pi}{4}.$$

$$\triangleright \ I_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \mathrm{d}\theta \int_0^{\frac{1}{\sin\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{1}{\sin^2\theta} = \frac{1}{2}\Big[-\cot\theta\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{1}{2}.$$

$$\triangleright \ I_3 = \int_{\frac{3\pi}{4}}^{\pi} \mathrm{d}\theta \int_0^{2\sin\theta} r\,\mathrm{d}r = 2\int_{\frac{3\pi}{4}}^{\pi} \sin^2\theta\,\mathrm{d}\theta = \int_{\frac{3\pi}{4}}^{\pi} (1 - \cos(2\theta))\,\mathrm{d}\theta$$

$$= \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\frac{3\pi}{4}}^{\pi} = \frac{\pi}{4} - \frac{1}{2}.$$

Hence, $I = I_1 + I_2 + I_3 = 1$.

Additional Question : Can you re-find the value of I geometrically ?

Exercise 6

1° (a) The considered differential equation is homogeneous. Infact,

$$y' = -\frac{y+2x}{x+2y} = -\frac{2+\dfrac{y}{x}}{1+2\dfrac{y}{x}}.$$

Let $y = zx$ then, $y' = xz' + z$. Thus the reduced differential equation, satisfied by the function $z = z(x)$ is

$$x\frac{\mathrm{d}z}{\mathrm{d}x} + z = -\frac{2+z}{1+2z},$$

and separating the variables, we find

$$\frac{1+2z}{1+z+z^2}\,\mathrm{d}z = -2\frac{\mathrm{d}x}{x}.$$

Its integration gives

$$\ln\left|2+2z+2z^2\right| = -2\ln|x| + c.$$

Finally, $\ln\left(2+2\dfrac{y}{x}+2\dfrac{y^2}{x^2}\right) = -2\ln|x| + c$, where c is an arbitrary constant.

(b) The considered differential equation is that with separable variables. We write,

$$\frac{\mathrm{d}y}{\sqrt{1-y}} = \frac{x\,\mathrm{d}x}{\sqrt{1+x^2}}.$$

By integration : $-2\sqrt{1-y} = \sqrt{1+x^2} + c$, with $c \in \mathbb{R}$.

2° The considered differential equation is linear of first order. Its general solution is looked for in the form $y = uv$. Upon substitution, we obtain,

$$u'v + u\left[v' - \frac{1+x}{x}v\right] = x^2.$$

The function v is chosen such that

$$v' - \frac{1+x}{x}v = 0 \qquad \text{that is} \qquad \frac{\mathrm{d}v}{v} = \left(\frac{1}{x}+1\right)\mathrm{d}x$$

that yields $\qquad v = xe^x.$

Besides,

$$u'v = x^2 \qquad \text{then} \qquad u' = xe^{-x}$$

therefore, $\quad u = -(1+x)e^{-x} + c.$
The general solution is thus,

$$y = \left[-1 - x + ce^{x}\right] x.$$

Finally, $y(-1) = 1$ gives $c = -e$.

LEBANESE UNIVERSITY Date: 10 February 2006
Faculty of Sciences (I)

Final (Group I) **Math171** Duration: 2 hours

ANSWER PAGE 57

Exercise 1

1° Give the f.e. of $(\cos x)^{\frac{1}{x^2}}$ to order 2, in neighborhood of 0.

2° Give the f.e. of $e^{\tan x}$ to order 3, in neighborhood of 0.

3° Give the f.e. of $\arctan \dfrac{1-x^2}{1+x^2}$ to order 12, in neighborhood of 0.

Exercise 2

Determine a and b such that

$$\lim_{x\to+\infty} x\left[2+(3+x)\ln\frac{x+a}{x+b}\right]=2.$$

Exercise 3

1° (a) Determine A, B, C and D such that

$$\frac{x}{(x+1)^2(1+x^2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}.$$

(b) Calculate $I=\displaystyle\int_0^1 \frac{x\,\mathrm{d}x}{(x+1)^2(1+x^2)}$.

2° Calculate $I=\displaystyle\int\sqrt{e^x-1}\,\mathrm{d}x$, setting $u=\sqrt{e^x-1}$.

Exercise 4

Let $I = \int_0^1 \mathrm{d}y \int_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x,y)\,\mathrm{d}x.$

1° Write down the equations of the boundary curves of the domain D, over which the double integral I is considered. Sketch D.

2° Rewrite the expression of the integral I by reversing the order of integration.

Exercise 5

Using polar coordinates, calculate

$$I = \iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2)^2}$$

with $D = \{(x,y)\ ;\ x^2+y^2-2x \leq 0,\ x \geq 1,\ y \geq 0\}$.

Exercise 1

1° We have $(\cos x)^{\frac{1}{x^2}} = e^{\frac{1}{x^2}\ln(\cos x)}$. In neighborhood of $x = 0$ we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4\varepsilon(x);$$

$$\begin{aligned}\ln(\cos x) &= \ln\left[1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right] + x^4\varepsilon(x)\\ &= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + x^4\varepsilon(x)\\ &= -\frac{x^2}{2} - \frac{x^4}{12} + x^4\varepsilon(x).\end{aligned}$$

Then, $(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}-\frac{x^2}{12}} + x^2\varepsilon(x) = \frac{1}{\sqrt{e}}\left(1 - \frac{x^2}{12}\right) + x^2\varepsilon(x)$.

2° In neighborhood of $x = 0$, $\tan x = x + \frac{x^3}{3} + x^3\varepsilon(x)$. Thus

$$\begin{aligned}e^{\tan x} &= 1 + x + \frac{x^3}{3} + \frac{\left(x + \frac{x^3}{3}\right)^2}{2!} + \frac{\left(x + \frac{x^3}{3}\right)^3}{3!} + x^3\varepsilon(x)\\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + x^3\varepsilon(x).\end{aligned}$$

3° Giving the f.e. of the derivative function of $f(x) = \arctan\frac{1-x^2}{1+x^2}$ to order 11, in neighborhood of 0,

$$f'(x) = \frac{\left(\frac{1-x^2}{1+x^2}\right)'}{1 + \left(\frac{1-x^2}{1+x^2}\right)^2} = \frac{-2x}{1+x^4} = -2x + 2x^5 - 2x^9 + x^{11}\varepsilon(x).$$

By integration we finally get

$$f(x) = f(0) - x^2 + \frac{x^6}{3} - \frac{x^{10}}{5} + x^{12}\varepsilon(x) = \frac{\pi}{4} - x^2 + \frac{x^6}{3} - \frac{x^{10}}{5} + x^{12}\varepsilon(x).$$

Exercise 2

Set $x = \dfrac{1}{t}$, $(x \to +\infty \Longleftrightarrow t \to 0_+)$. Hence,

$$f\left(\frac{1}{t}\right) = \frac{1}{t}\left[2 + \frac{1+3t}{t}\ln\left(\frac{1+at}{1+bt}\right)\right].$$

In neighborhood of $t = 0$ and to order 2, we have

$$\ln\frac{1+at}{1+bt} = \ln(1+at) - \ln(1+bt) = (a-b)t - \frac{1}{2}(a^2-b^2)t^2 + t^2\varepsilon(t).$$

It follows that

$$f\left(\frac{1}{t}\right) = \frac{2+a-b}{t} + \frac{b^2-a^2}{2} + 3(a-b) + \varepsilon(t).$$

The limit of $f(x)$ as x tends to $+\infty$ would be equal to 2, iff

$$2 + a - b = 0; \qquad \text{and}$$
$$\frac{b^2-a^2}{2} + 3(a-b) = 2$$

which is equivalent to,

$$\begin{cases} a - b = -2; \\ 3(a-b) - \dfrac{1}{2}(a+b)(a-b) = 2 \end{cases} \quad \text{that is,} \quad \begin{cases} a - b = -2; \\ a + b = 8. \end{cases}$$

Hence, $a = 3$ and $b = 5$.

Exercise 3

1° (a) $\dfrac{x}{(x+1)^2(1+x^2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{Cx+D}{x^2+1}.$

Reducing to same denominator, and then by identification, the unknowns A, B, C and D verify the following system of equations

$$\begin{cases} A + C = 0; \\ A + B + 2C + D = 0; \\ A + C + 2D = 1; \\ A + B + D = 0, \end{cases} \quad \text{then} \quad \begin{cases} A = C = 0; \\ B = -\dfrac{1}{2}; \\ D = \dfrac{1}{2}. \end{cases}$$

(b) $$I = -\frac{1}{2}\int_0^1 \frac{\mathrm{d}x}{(1+x)^2} + \frac{1}{2}\int_0^1 \frac{\mathrm{d}x}{1+x^2}$$
$$= \frac{1}{2}\left[\frac{1}{1+x}\right]_0^1 + \frac{1}{2}\left[\arctan x\right]_0^1 = \frac{\pi}{8} - \frac{1}{4}.$$

2° With $u = \sqrt{e^x - 1}$, it follows that,

$$x = \ln(1+u^2) \qquad \text{and} \qquad \mathrm{d}x = \frac{2u}{1+u^2}\,\mathrm{d}u.$$

Hence,

$$I = 2\int \frac{u^2}{1+u^2}\,\mathrm{d}u = 2\int \left(1 - \frac{1}{1+u^2}\right)\mathrm{d}u = 2\,(u - \arctan u) + \text{cnst}.$$

Retaining x back we get :

$$I = 2\left[\sqrt{e^x - 1} - \arctan\sqrt{e^x - 1}\right] + \text{cnst}.$$

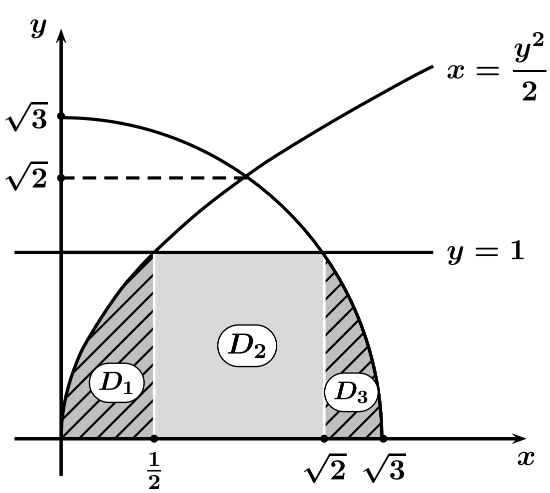
Exercise 4

$D = D_1 \cup D_2 \cup D_3$, where

$$D_1 : \begin{cases} 0 \le x \le \frac{1}{2} \\ 0 \le y \le \sqrt{2x} \end{cases}$$

$$D_2 : \begin{cases} \frac{1}{2} \le x \le \sqrt{2} \\ 0 \le y \le 1 \end{cases}$$

$$D_3 : \begin{cases} \sqrt{2} \le x \le \sqrt{3} \\ 0 \le y \le \sqrt{3 - x^2}. \end{cases}$$

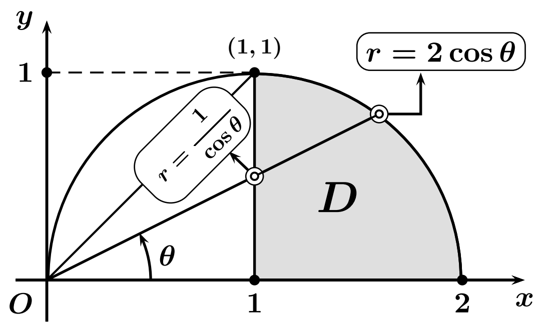


$$I = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_3} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^{\frac{1}{2}}\mathrm{d}x\int_0^{\sqrt{2x}} f(x,y)\,\mathrm{d}y + \int_{\frac{1}{2}}^{\sqrt{2}}\mathrm{d}x\int_0^1 f(x,y)\,\mathrm{d}y + \int_{\sqrt{2}}^{\sqrt{3}}\mathrm{d}x\int_0^{\sqrt{3-x^2}} f(x,y)\,\mathrm{d}y.$$

Exercise 5

The domain D is defined in polar coordinates as

$$D : \begin{cases} 0 \le \theta \le \dfrac{\pi}{4}; \\ \dfrac{1}{\cos\theta} \le r \le 2\cos\theta. \end{cases}$$



The integral I is written as

$$I = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{r\,\mathrm{d}r}{r^4}.$$

Hence,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\frac{1}{\cos\theta}}^{2\cos\theta} \frac{\mathrm{d}r}{r^3} = -\frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\frac{1}{r^2}\right]_{\frac{1}{\cos\theta}}^{2\cos\theta} \mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos^2\theta - \frac{1}{4\cos^2\theta}\right) \mathrm{d}\theta. \end{aligned}$$

In fact,

$$\int_0^{\frac{\pi}{4}} \cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(2\theta))\,\mathrm{d}\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{1}{4};$$

$$\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\cos^2\theta} = \Big[\tan\theta\Big]_0^{\frac{\pi}{4}} = 1.$$

Therefore, $\quad I = \dfrac{\pi}{16}.$

LEBANESE UNIVERSITY Date: 10 February 2006

Faculty of Sciences (I)

Final (Group II) **Math171** **Duration: 2 hours**

ANSWER PAGE 63

Exercise 1

1° Give the f.e. of $\dfrac{e^x - e^{-x}}{\sin x}$ to order 4, in neighborhood of 0.

2° Give the f.e. of $e^{\sqrt{1+x}}$ to order 3, in neighborhood of 0.

3° Give the f.e. of $\arcsin x$ to order 3, in neighborhood of $x = \dfrac{1}{2}$.

Exercise 2

Let $f(x) = \ln \dfrac{1+\sin x}{1-\sin x}$.

1° Give the finite expansion of order 3 of f in neighborhood of 0.

2° Calculate $f'(0)$.

3° Determine the equation of the tangent Δ to the representative curve C of f at point $O(0,0)$. Precise the position of Δ relative to C, in neighborhood of 0.

Exercise 3

1° (a) Determine A, B and C such that

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

(b) Calculate $I = \displaystyle\int \frac{\mathrm{d}x}{x^3-1}$.

2° Calculate $I_1 = \displaystyle\int \frac{\mathrm{d}x}{(x+1)\sqrt{x}}$ and $I_2 = \displaystyle\int_{-1}^{1} (\arccos x)^2 \,\mathrm{d}x$.

Exercise 4

Let $I = \displaystyle\int_{-6}^{2} \mathrm{d}x \int_{\frac{x^2}{4}-1}^{2-x} f(x,y)\,\mathrm{d}y$.

1° Write down the equations of the boundary curves of the domain D, over which the double integral I is considered. Sketch D.

2° Rewrite the expression of I after reversing the order of integration.

Exercise 5

Use polar coordinates to calculate

$$I = \iint_D \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2+y^2)^2}$$

with $\quad D = \{(x, y)\ ;\ 0 \le x \le x^2 + y^2 \le 1,\ y \ge 0\}.$

LU Sciences (I) Year 2005-2006	Solution	Calculus Math171 (Group II)

Exercise 1

1° $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + x^5\varepsilon(x);$

$e^{-x} = 1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dfrac{x^4}{24} - \dfrac{x^5}{120} + x^5\varepsilon(x);$

$\sin x = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} + x^5\varepsilon(x).$

Then, $\dfrac{e^x - e^{-x}}{\sin x} = \dfrac{2 + \dfrac{x^2}{3} + \dfrac{x^4}{60}}{1 - \dfrac{x^2}{6} + \dfrac{x^4}{120}} + x^4\varepsilon(x) = 2 + \dfrac{2x^2}{3} + \dfrac{x^4}{9} + x^4\varepsilon(x).$

2° $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} + x^3\varepsilon(x)$. Thus

$$\begin{aligned} e^{\sqrt{1+x}} &= e^{1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}} + x^3\varepsilon(x) \\ &= e\left[1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \frac{\left(\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)^2}{2!} + \frac{\left(\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\right)^3}{3!}\right] + x^3\varepsilon(x) \\ &= e\left(1 + \frac{x}{2} + \frac{x^3}{48}\right) + x^3\varepsilon(x). \end{aligned}$$

3° Set $x = \dfrac{1}{2} + t$. To order 2, and in neighborhood of $t = 0$, we have

$$\begin{aligned} \arcsin'\left(t + \frac{1}{2}\right) = \frac{1}{\sqrt{\frac{3}{4} - t - t^2}} &= \frac{2}{\sqrt{3}}\left[1 + \left(-\frac{4}{3}(t + t^2)\right)\right]^{-\frac{1}{2}} \\ &= \frac{2}{\sqrt{3}}\left[1 + \frac{2}{3}(t + t^2) + \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)}{2!}\left(-\frac{4}{3}(t + t^2)\right)^2\right] + t^2\varepsilon(t) \\ &= \frac{2}{\sqrt{3}}\left(1 + \frac{2t}{3} + \frac{4t^2}{3}\right) + t^2\varepsilon(t). \end{aligned}$$

By Integration : $\arcsin\left(t + \dfrac{1}{2}\right) = \arcsin\dfrac{1}{2} + \dfrac{2t}{\sqrt{3}} + \dfrac{2t^2}{3\sqrt{3}} + \dfrac{8t^3}{9\sqrt{3}} + t^3\varepsilon(t).$

Then finally :

$$\arcsin x = \frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right) + \frac{2}{3\sqrt{3}}\left(x - \frac{1}{2}\right)^2 + \frac{8}{9\sqrt{3}}\left(x - \frac{1}{2}\right)^3 + \left(x - \frac{1}{2}\right)^3\varepsilon(x).$$

Exercise 2

1° Expanding in neighborhood of 0 up to order 3, we get

$$\begin{aligned}
\ln(1+\sin x) &= \ln\left[1+\left(x-\frac{x^3}{6}\right)\right]+x^3\varepsilon(x)\\
&= x-\frac{x^3}{6}-\frac{\left(x-\frac{x^3}{6}\right)^2}{2}+\frac{\left(x-\frac{x^3}{6}\right)^3}{3}+x^3\varepsilon(x)\\
&= x-\frac{x^2}{2}+\frac{x^3}{6}+x^3\varepsilon(x);\\
\ln(1-\sin x) &= \ln\left[1+\left(-x+\frac{x^3}{6}\right)\right]+x^3\varepsilon(x)\\
&= -x+\frac{x^3}{6}-\frac{\left(-x+\frac{x^3}{6}\right)^2}{2}+\frac{\left(-x+\frac{x^3}{6}\right)^3}{3}+x^3\varepsilon(x)\\
&= -x-\frac{x^2}{2}-\frac{x^3}{6}+x^3\varepsilon(x).
\end{aligned}$$

It implies that

$$f(x)=\ln\frac{1+\sin x}{1-\sin x}=2x+\frac{x^3}{3}+x^3\varepsilon(x).$$

2° $f'(0)=2$ (it is the coefficient of x in the f.e.).

3° The equation of the tangent Δ to the curve C of f at point $O(0,0)$ is $y=2x$. Moreover, in neighborhood of 0, $f(x)-2x\simeq\dfrac{x^3}{6}$. Thus,

- ▷ As $x\to 0_+$, C is above Δ ;
- ▷ As $x\to 0_-$, C is below Δ.

Exercise 3

1° (a) $$\frac{1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\Rightarrow\begin{cases}A+B=0;\\A-C=1;\\A-B+C=0,\end{cases}\Rightarrow\begin{cases}A=\frac{1}{3};\\B=-\frac{1}{3};\\C=-\frac{2}{3}.\end{cases}$$

(b) $I = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,\mathrm{d}x$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x - \frac{1}{2}\int \frac{1}{x^2+x+1}\,\mathrm{d}x$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2}\int \frac{1}{\frac{3}{4}+\left(x+\frac{1}{2}\right)^2}\,\mathrm{d}x$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \text{cnst.}$$

2° **Evaluation of I_1 :** Performing the change of variable $t = \sqrt{x}$. It follows that

$$I_1 = \int \frac{2\,\mathrm{d}t}{1+t^2} = 2\arctan t + \text{cnst} = 2\arctan\sqrt{x} + \text{cnst.}$$

Evaluation of I_2 : A first integration by parts gives

$$I_2 = \left[x\,(\arccos x)^2\right]_{-1}^{1} + 2\int_{-1}^{1} \arccos x \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$$

$$= \pi^2 - 2\int_{-1}^{1} \arccos x\,\mathrm{d}\left(\sqrt{1-x^2}\right).$$

Integrating by parts once more, we get

$$I_2 = \pi^2 - 2\left(\left[\sqrt{1-x^2}\arccos x\right]_{-1}^{1} + \int_{-1}^{1}\mathrm{d}x\right) = \pi^2 - 4.$$

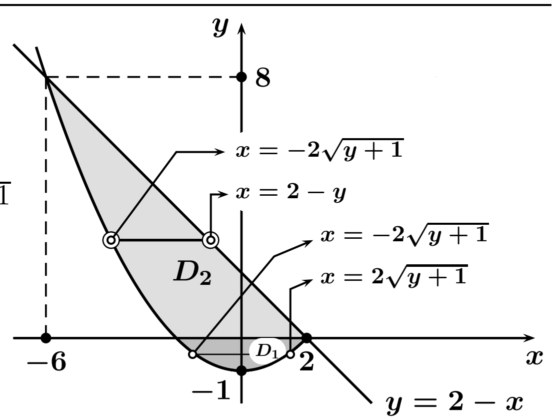
Exercise 4

We have

$D = D_1 \cup D_2$

$$D_1 : \begin{cases} -1 \le y \le 0 \\ -2\sqrt{y+1} \le x \le 2\sqrt{y+1} \end{cases}$$

$$D_2 : \begin{cases} 0 \le y \le 8 \\ -2\sqrt{y+1} \le x \le 2 - y \end{cases}$$



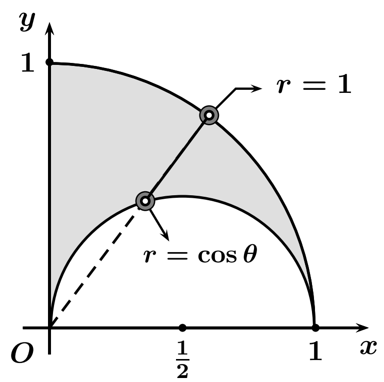
The integral I is written as

$$I = \int_{-1}^{0}\mathrm{d}y\int_{-2\sqrt{y+1}}^{2\sqrt{y+1}} f(x,y)\,\mathrm{d}x + \int_{0}^{8}\mathrm{d}y\int_{-2\sqrt{y+1}}^{2-y} f(x,y)\,\mathrm{d}x.$$

Exercise 5

The domain D is defined in polar coordinates as

$$D : \begin{cases} 0 \le \theta \le \dfrac{\pi}{2}; \\ \cos\theta \le r \le 1. \end{cases}$$



The integral I is written in the form

$$I = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\cos\theta}^{1} \frac{r\,\mathrm{d}r}{(1+r^2)^2}.$$

Thus,

$$\begin{aligned} I &= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{1+r^2}\right]_{\cos\theta}^{1} \mathrm{d}\theta = -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{1+\cos^2\theta}\right) \mathrm{d}\theta \\ &= -\frac{\pi}{8} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{1+\cos^2\theta}. \end{aligned}$$

And,

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{1+\cos^2\theta} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}(\tan\theta)}{2+\tan^2\theta} = \left[\frac{1}{\sqrt{2}}\arctan\frac{\tan\theta}{\sqrt{2}}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2\sqrt{2}}.$$

Therefore finally, $\quad I = \dfrac{\pi}{8}(\sqrt{2}-1).$

LEBANESE UNIVERSITY Date: March 2007

Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 69

Exercise 1 **[15 pts]**

1° Give the f.e. of order 3 of $f(x) = \dfrac{\ln(\cos x)}{\cos x - 1}$ in neighborhood of $x = 0$.

2° Let f be a function defined by $f(x) = \dfrac{e^{-x}\sin x}{x}$.

(a) Write down the f.e. of f to order 2, in neighborhood of $x = 0$. Show that f is extendable by continuity to $x = 0$, with function g as its extension .

(b) Determine the tangent to the curve of g at the point of abscissa 0, as well as its position relative to that curve near that point.

Exercise 2 **[15 pts]**

1° Let $J = \displaystyle\int_0^{\frac{\sqrt{2}}{2}} \frac{u^2}{1-u^2}\,\mathrm{d}u$.

(a) Find the real numbers a, b and c such that

$$\frac{u^2}{1-u^2} = \frac{a}{1-u} + \frac{b}{1+u} + c.$$

(b) Deduce that $J = \dfrac{1}{2}\ln\left(3 + 2\sqrt{2}\right) - \dfrac{\sqrt{2}}{2}$.

2° Calculate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos x}\,\mathrm{d}x$, setting $u = \sin x$.

3° Calculate $K = \displaystyle\int_0^{\frac{\pi}{4}} \cos(x)\ln(\cos x)\,\mathrm{d}x$.

Exercise 3 **[10 pts]**

Choose and treat **one and only one** of the following questions :

1° Calculate $I(x) = \displaystyle\int \frac{\sin x \tan x}{\cos^3 x + \sin^3 x}\,\mathrm{d}x$, and $J(x) = \displaystyle\int \frac{1}{x^2(x+1)}\,\mathrm{d}x$.

2° Solve the differential equation : $y' - \dfrac{1}{x(1+x)}y = \dfrac{1}{1+x}$.

Exercise 4 ______________________________ **[15 pts]**

The aim of this exercise is to evaluate the integral $\boldsymbol{I = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos x)}{\cos x}\,\mathrm{d}x}$**.**

1° (a) Let $a \in \mathbb{R}$ and $b \in [0, 1[$. Setting $t = \tan \frac{x}{2}$, calculate

$$J = \int_0^{\frac{\pi}{2}} \frac{a}{1 + b\cos x}\,\mathrm{d}x.$$

(b) Deduce the equality $\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y \cos x}\,\mathrm{d}x = y.$

2° Calculate the double integral

$$K = \iint_D \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}x\,\mathrm{d}y$$

where D is the square $\left\{(x, y) \in \mathbb{R}^2 : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le \frac{\pi}{2}\right\}.$

3° Choosing an appropriate order of integration for K, deduce the value of I.

Exercise 5 ______________________________ **[15 pts]**

Consider the double integral

$$I = \int_0^{\sqrt{3}} \mathrm{d}y \int_{y\sqrt{3}}^{3} e^{x^2}\,\mathrm{d}x.$$

1° Represent graphically the domain of integration of I.

2° Calculate I by altering the order of integration.

3° Using polar coordinates, calculate I.

Exercise 1

1° Expanding to order 5 in neighborhood of 0, we get

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^5\varepsilon(x);$$

$$\begin{aligned}\ln(\cos x) &= \ln\left[1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right] + x^5\varepsilon(x) \\ &= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + x^5\varepsilon(x) = -\frac{x^2}{2} - \frac{x^4}{12} + x^5\varepsilon(x).\end{aligned}$$

Thus, $f(x) = -\dfrac{\frac{1}{2} + \frac{x^2}{12}}{-\frac{1}{2} + \frac{x^2}{24}} + x^3\varepsilon(x) = 1 + \dfrac{x^2}{4} + x^3\varepsilon(x).$

2° (a) In neighborhood of $x = 0$

$$\sin x = x - \frac{x^3}{6} + x^3\varepsilon(x), \quad \text{and} \quad e^{-x} = 1 - x + \frac{x^2}{2} + x^2\varepsilon(x).$$

From which follows $f(x) = 1 - x + \dfrac{x^2}{3} + x^2\varepsilon(x)$. As, $\lim\limits_{x\to 0} f(x) = 1$, then f is extendable by continuity to 0, with extension function g, defined by : $g(x) = f(x)$ if $x \neq 0$ and $g(0) = 1$.

(b) The equation of the tangent (D) to the representative curve (C) of g, at the point of abscissa 0 is $y = 1 - x$. Moreover, (C) is above (D) in neighborhood of $x = 0$, since $f(x) - y \underset{0}{\simeq} \dfrac{x^2}{3} > 0$.

Exercise 2

1° (a) We have

$$\frac{a}{1-u} + \frac{b}{1+u} + c = \frac{a+b+c+(a-b)u - cu^2}{1-u^2}.$$

Thus by identification, we have

$$\frac{a}{1-u} + \frac{b}{1+u} + c = \frac{a+b+c+(a-b)u - cu^2}{1-u^2} \equiv \frac{u^2}{1-u^2},$$

iff $c = -1$, $a - b = 0$ and $a + b + c = 0$. Solving the system, we get $c = -1$, $a = b = \dfrac{1}{2}$.

(b) From question 1°, it is deduced that

$$\begin{aligned} J &= \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1-u} + \frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1+u} - \int_0^{\frac{\sqrt{2}}{2}} \mathrm{d}u \\ &= \left[-\frac{1}{2}\ln|1-u|\right]_0^{\frac{\sqrt{2}}{2}} + \left[\frac{1}{2}\ln|1+u|\right]_0^{\frac{\sqrt{2}}{2}} - \frac{\sqrt{2}}{2} \\ &= \frac{1}{2}\ln\left(3+2\sqrt{2}\right) - \frac{\sqrt{2}}{2}. \end{aligned}$$

2° By means of the change of variable $u = \sin x$, we notice that

$$M = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos x}\,\mathrm{d}x = \int_0^{\frac{\sqrt{2}}{2}} \frac{u^2}{1-u^2}\,\mathrm{d}u = J.$$

3° Integrating by parts with $u = \ln(\cos x)$ and $\mathrm{d}v = \cos x\,\mathrm{d}x$, we get

$$K = \Big[\sin x \ln(\cos x)\Big]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos x}\mathrm{d}x = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) - \frac{\sqrt{2}}{4}\ln 2 - \frac{\sqrt{2}}{2}.$$

Exercise 3

1° • $I(x) = \displaystyle\int \frac{\sin x \tan x}{\cos^3 x + \sin^3 x}\,\mathrm{d}x = \int \frac{\tan^2 x\,\mathrm{d}(\tan x)}{1+\tan^3 x} \underset{t=\tan x}{=} \int \frac{t^2\,\mathrm{d}t}{1+t^3}.$

From which it follows that $I = \dfrac{1}{3}\ln\left|1+\tan^3 x\right| + \text{cnst}.$

• $J(x) = \displaystyle\int \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{1+x}\right)\mathrm{d}x = -\frac{1}{x} - \ln|x| + \ln|1+x| + \text{cnst}.$

2° It is a linear differential equation of first order. The general solution y of this equation is sought for in the form $y(x) = u(x)v(x)$. Then $y' = u'v + v'u$, and the considered diff. equation is rewritten as :

$$u'v + u\left[v' - \frac{1}{x(1+x)}v\right] = \frac{1}{1+x}.$$

► $v' - \dfrac{1}{x(x+1)}v = 0$, which implies $v(x) = \dfrac{x}{1+x}$.

► $u'v = \dfrac{1}{1+x}$, yields $u(x) = \ln|x| + k$.

Thus, $y = (\ln|x| + k)\dfrac{x}{1+x}$, where k is an arbitrary constant.

Exercise 4

1° (a) For $t = \tan \frac{x}{2}$, we have $\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$. It follows that

$$J = 2a\int_0^1 \frac{\mathrm{d}t}{(1+b)+(1-b)t^2} = \frac{2a}{\sqrt{1-b^2}} \arctan\sqrt{\frac{1-b}{1+b}}.$$

(b) Using the result of the preceding question with $a = \sin y$ and $b = \cos y$ for $y \in \left]0, \frac{\pi}{2}\right]$, we have

$$\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}x = \frac{2\sin y}{\sqrt{1-\cos^2 y}} \arctan\sqrt{\frac{1-\cos y}{1+\cos y}}.$$

And since $\frac{2\sin y}{\sqrt{1-\cos^2 y}} = 2$, then :

$$\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}x = 2\arctan\sqrt{\frac{1-\cos y}{1+\cos y}}.$$

Moreover, it is easily verified that $\frac{1-\cos y}{1+\cos y} = \tan^2 \frac{y}{2}$, and consequently,

$$\arctan\sqrt{\frac{1-\cos y}{1+\cos y}} = \arctan\left(\tan\frac{y}{2}\right) = \frac{y}{2}.$$

Then finally : $\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}x = y.$

2° Integrating first with respect to x, and further with respect to y, we get

$$K = \int_0^{\frac{\pi}{2}} \mathrm{d}y\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}x \underset{\text{From } 1°(b)}{=} \int_0^{\frac{\pi}{2}} y\,\mathrm{d}y = \frac{\pi^2}{8}.$$

3° Integrating first with respect to y, and further with respect to x, we get

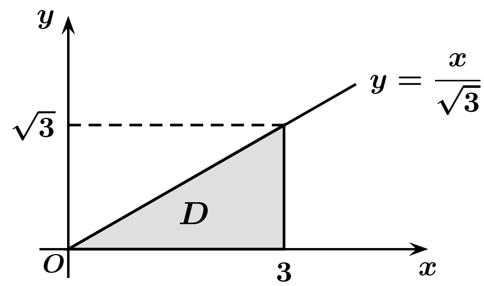
$$K = \int_0^{\frac{\pi}{2}} \mathrm{d}x\int_0^{\frac{\pi}{2}} \frac{\sin y}{1+\cos y\cos x}\,\mathrm{d}y = -\int_0^{\frac{\pi}{2}} \left[\frac{1}{\cos x}\ln(1+\cos y\cos x)\right]_{y=0}^{y=\frac{\pi}{2}} \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos x)}{\cos x}\,\mathrm{d}x.$$

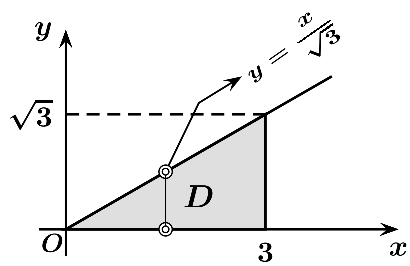
It yields $I = \int_0^{\frac{\pi}{2}} \frac{\ln(1+\cos x)}{\cos x}\,\mathrm{d}x = K = \frac{\pi^2}{8}.$

Exercise 5

1° The domain D is represented in the following figure :



2° Changing the order of integration, the domain D is defined as



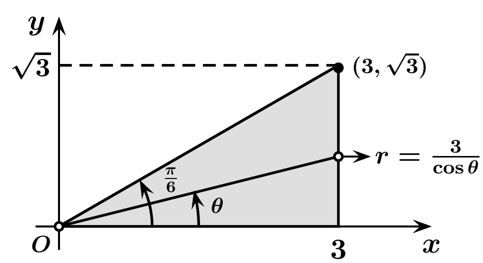
$$D : \begin{cases} 0 \le x \le 3; \\ 0 \le y \le \dfrac{x}{\sqrt{3}}. \end{cases}$$

Then

$$I = \int_0^3 \mathrm{d}x \int_0^{\frac{x}{\sqrt{3}}} e^{x^2}\,\mathrm{d}y = \frac{1}{2\sqrt{3}} \int_0^3 e^{x^2}\,\mathrm{d}(x^2) = \frac{\sqrt{3}}{6}\left(e^9 - 1\right).$$

3° In polar coordinates, D is defined as

$$D : \begin{cases} 0 \le \theta \le \dfrac{\pi}{6}; \\ 0 \le r \le \dfrac{3}{\cos\theta}. \end{cases}$$



Then,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \int_0^{\frac{3}{\cos\theta}} e^{r^2\cos^2\theta} r\,\mathrm{d}r \\ &= \int_0^{\frac{\pi}{6}} \frac{1}{2\cos^2\theta}\left[e^{r^2\cos^2\theta}\right]_{r=0}^{r=\frac{3}{\cos\theta}} \mathrm{d}\theta = \frac{e^9-1}{2}\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}\theta}{\cos^2\theta}. \end{aligned}$$

It follows that : $I = \dfrac{e^9-1}{2}\tan\dfrac{\pi}{6} = \dfrac{e^9-1}{6}\sqrt{3}.$

LEBANESE UNIVERSITY Date: 22 Feb. 2008

Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 75

Exercise 1 **[22 pts]**

1° Let $f(x) = \dfrac{\ln(3-2x)}{x}$.

(a) Write down the f.e. of order 3 of f in neighborhood of $x = 1$.

(b) Calculate $f'(1)$.

(c) Determine the equation of the tangent Δ to the representative curve Γ of f, at the point abscissa $x = 1$, and precise the relative position of Γ and Δ in neighborhood of $x = 1$.

2° Calculate $\lim\limits_{x\to 0} \dfrac{e^{\frac{\sin x}{x}} - e^{\frac{x}{\sinh x}}}{x^4}$.

3° Determine the equation of the asymptote T at $+\infty$ to the curve C of

$$f(x) = x\left(\sqrt[3]{x^3+3x^2+1} - \sqrt{x^2-2x+2}\right)$$

and precise the relative positions of T and C in neighborhood of $+\infty$.

Exercise 2 **[10 pts]**

Calculate the following integrals

$$I(x) = \int \frac{2x+2}{x^2+x+1}\,\mathrm{d}x, \quad J(x) = \int x^2 \ln(1+x^2)\,\mathrm{d}x, \quad K(x) = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,\mathrm{d}x.$$

Exercise 3 **[14 pts]**

1° Calculate $I(t) = \displaystyle\int \frac{1}{(1-t^2)t^2}\,\mathrm{d}t$.

2° Deduce $J(\theta) = \displaystyle\int \frac{\mathrm{d}\theta}{\cos\theta \sin^2\theta}$.

3° Calculate $K(x) = \displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\,\mathrm{d}x$.

Exercise 4 **[24 pts]**

1° Calculate $A = \int_0^1 \mathrm{d}y \int_{\sqrt{y}}^1 \sin(x^3)\,\mathrm{d}x$.

2° Let $B = \int_0^3 \mathrm{d}y \int_{-3\sqrt{y}}^{\sqrt{y}} f(x,y)\,\mathrm{d}x$.

(a) Change the order of integration of B.

(b) Write down the expression of B by using polar coordinates.

Exercise 1

1° (a) Let $x = 1 + t$, and expanding in neighborhood of $t = 0$, we have

$$f(x) = \frac{\ln(1-2t)}{1+t} = -2t - \frac{8}{3}t^3 + t^3\varepsilon(t).$$

Then the f.e. of order 3 of $f(x)$ in neighborhood of $x = 1$ is

$$f(x) = -2(x-1) - \frac{8}{3}(x-1)^3 + (x-1)^3\varepsilon(x).$$

(b) $f'(1) = -2$ (it is the coefficient of $(x-1)$ in the f.e.).

(c) The equation of the tangent at the point $(1, 0)$ is : $y = -2(x-1)$.

- ► $f(x) - y \simeq -\frac{8}{3}(x-1)^3 < 0$, as $x \to 1_+$, hence the curve is below the tangent.
- ► $f(x) - y \simeq -\frac{8}{3}(x-1)^3 > 0$, as $x \to 1_-$, hence the curve is above the tangent.

2° In neighborhood of 0, and up to order 5, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x) \quad \text{and} \quad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x).$$

Thus, in neighborhood of 0, and up to order 4, we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + x^4\varepsilon(x) \quad \text{and} \quad \frac{x}{\sinh x} = 1 - \frac{x^2}{6} + \frac{7x^4}{360} + x^4\varepsilon(x).$$

Expanding $e^{\frac{\sin x}{x}}$ and $e^{\frac{x}{\sinh x}}$ in neighborhood of 0, and to order 4, we have

$$\begin{aligned} e^{\frac{\sin x}{x}} &= e^1 \times e^{-\frac{x^2}{6} + \frac{x^4}{120}} + x^4\varepsilon(x) \\ &= e\left(1 + \left(-\frac{x^2}{6} + \frac{x^4}{120}\right) + \frac{1}{2}\left(-\frac{x^2}{6} + \frac{x^4}{120}\right)^2\right) + x^4\varepsilon(x) \\ &= e\left(1 - \frac{x^2}{6} + \frac{x^4}{45}\right) + x^4\varepsilon(x). \end{aligned}$$

We find as well, $e^{\frac{x}{\sinh x}} = e\left(1 - \frac{x^2}{6} + \frac{x^4}{30}\right) + x^4\varepsilon(x)$. It follows that,

$$\lim_{x\to 0} \frac{e^{\frac{\sin x}{x}} - e^{\frac{x}{\sinh x}}}{x^4} = \lim_{x\to 0} \frac{e\left(1 - \frac{x^2}{6} + \frac{x^4}{45}\right) - e\left(1 - \frac{x^2}{6} + \frac{x^4}{30}\right) + x^4\varepsilon(x)}{x^4} = -\frac{e}{90}.$$

3° Let $x = \dfrac{1}{t}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. We begin by expanding finitely $tf\left(\dfrac{1}{t}\right)$ to order 2 in neighborhood of $t = 0$,

$$tf\left(\frac{1}{t}\right) = \frac{\sqrt[3]{1+3t+t^3} - \sqrt{1-2t+2t^2}}{t}.$$

In neighborhood of $t = 0$, and up to order 3, we have

$$\begin{aligned}\sqrt[3]{1+3t+t^3} &= 1 + \frac{1}{3}(3t+t^3) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(3t+t^3)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(3t+t^3)^3 + t^3\varepsilon(t)\\ &= 1 + t - t^2 + 2t^3 + t^3\varepsilon(t).\end{aligned}$$

And,

$$\begin{aligned}\sqrt{1-2t+2t^2} &= 1 + \frac{1}{2}(-2t+2t^2) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2t+2t^2)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-2t+2t^2)^3 + t^3\varepsilon(t)\\ &= 1 - t + \frac{1}{2}t^2 + \frac{1}{2}t^3 + t^3\varepsilon(t).\end{aligned}$$

Then

$$tf\left(\frac{1}{t}\right) = 2 - \frac{3}{2}t + \frac{3}{2}t^2 + t^2\varepsilon(t).$$

Thus, as $x \to +\infty$,

$$f(x) = 2x - \frac{3}{2} + \frac{3}{2x} + \frac{1}{x}\varepsilon(x).$$

$y = 2x - \dfrac{3}{2}$ is then the equation of the oblique asymptote to the infinite branch $(+\infty)$ of the representative curve of the function f. Moreover, $f(x) - y \underset{+\infty}{\simeq} \dfrac{3}{2x} > 0$ so there the curve is above its asymptote.

Exercise 2

Evaluation of $I(x)$:

$$\begin{aligned}I(x) &= \int \frac{(2x+1)+1}{x^2+x+1}\,\mathrm{d}x\\ &= \int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}.\end{aligned}$$

Thus, $\quad I(x) = \ln(x^2+x+1) + \dfrac{2}{\sqrt{3}}\arctan\left(\dfrac{2x+1}{\sqrt{3}}\right) + c$, where $c \in \mathbb{R}$.

Evaluation of $J(x)$: To integrate by parts, set

$$\left\{\begin{array}{rcl} u &=& \ln(1+x^2) \\ \mathrm{d}v &=& x^2\,\mathrm{d}x \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \mathrm{d}u &=& \dfrac{2x\,\mathrm{d}x}{1+x^2}; \\ v &=& \dfrac{x^3}{3}. \end{array}\right.$$

Hence it yields,

$$J(x) = uv - \int v\,\mathrm{d}u = \frac{x^3}{3}\ln(1+x^2) - \frac{2}{3}\int \frac{x^4}{1+x^2}\,\mathrm{d}x.$$

Since, $\dfrac{x^4}{1+x^2} = x^2 - 1 + \dfrac{1}{1+x^2}$, then $\displaystyle\int \frac{x^4}{1+x^2}\,\mathrm{d}x = \frac{x^3}{3} - x + \arctan x + c_1$, where $c_1 \in \mathbb{R}$.

Then : $J(x) = \dfrac{x^3}{3}\ln(1+x^2) - \dfrac{2x^3}{9} + \dfrac{2x}{3} - \dfrac{2}{3}\arctan x + c_2$, where $c_2 \in \mathbb{R}$.

Evaluation of $K(x)$: We have $K(x) = \ln(e^x + e^{-x}) + c$, where $c \in \mathbb{R}$.

Exercise 3

1° $I(t) = \displaystyle\int \left(\frac{1}{1-t^2} + \frac{1}{t^2}\right)\mathrm{d}t = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| - \frac{1}{t} + c$, where $c \in \mathbb{R}$.

2° Setting $t = \sin\theta$, then

$$J(\theta) = \int \frac{\cos\theta\,\mathrm{d}\theta}{\cos^2\theta\sin^2\theta} = \int \frac{\cos\theta\,\mathrm{d}\theta}{(1-\sin^2\theta)\sin^2\theta} = I(t).$$

Thus, $\quad J(\theta) = \dfrac{1}{2}\ln\dfrac{1+\sin\theta}{1-\sin\theta} - \dfrac{1}{\sin\theta} + c.$

3° Put $x = \tan\theta$ (more precisely $\theta = \arctan x$ with $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$). Then,

$$K(x) = \int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\frac{\mathrm{d}\theta}{\cos^2\theta} = \int \frac{\mathrm{d}\theta}{\cos\theta\sin^2\theta} = J(\theta).$$

Remark : We may integrate by parts, setting

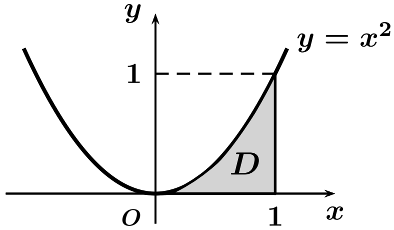
$$\left\{\begin{array}{rcl} u &=& \sqrt{1+x^2} \\ \mathrm{d}v &=& \dfrac{\mathrm{d}x}{x^2} \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} \mathrm{d}u &=& \dfrac{x\,\mathrm{d}x}{\sqrt{1+x^2}}; \\ v &=& -\dfrac{1}{x}. \end{array}\right.$$

Then, we have

$$J(x) = -\frac{\sqrt{1+x^2}}{x} + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = -\frac{\sqrt{1+x^2}}{x} + \ln(x + \sqrt{x^2+1}) + c.$$

Exercise 4

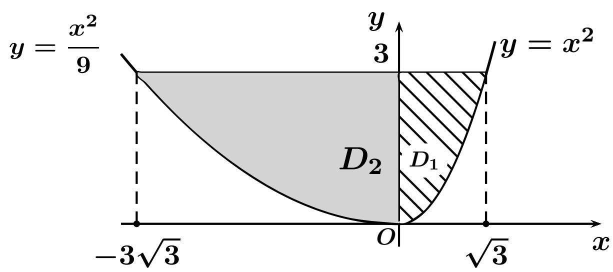
1° The domain D is represented by the following figure.



Changing the order of integration, the integral I is written as

$$\begin{aligned} A &= \int_0^1 \mathrm{d}x \int_0^{x^2} \sin(x^3)\,\mathrm{d}y \\ &= \int_0^1 x^2 \sin(x^3)\,\mathrm{d}x = -\frac{\cos(x^3)}{3}\Big]_0^1 \\ &= \frac{1}{3}(1-\cos 1). \end{aligned}$$

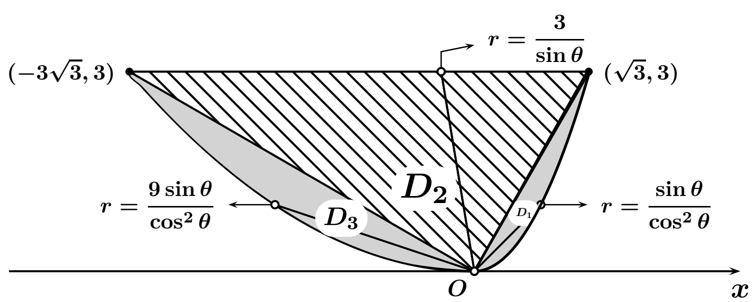
2° (a) The domain D is represented by the following figure.



Changing the order of integration, the integral B is written as

$$B = \int_{-3\sqrt{3}}^{0} \mathrm{d}x \int_{\frac{x^2}{9}}^{3} f(x,y)\,\mathrm{d}y + \int_0^{\sqrt{3}} \mathrm{d}x \int_{x^2}^{3} f(x,y)\,\mathrm{d}y.$$

(b) In polar coordinates, D is partitioned in the following manner



By means of polar coordinates,the integral B is written as

$$\begin{aligned} B = &\int_0^{\frac{\pi}{3}} \mathrm{d}\theta \int_0^{\frac{\sin\theta}{\cos^2\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r + \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \mathrm{d}\theta \int_0^{\frac{3}{\sin\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r \\ &+ \int_{\frac{5\pi}{6}}^{\pi} \mathrm{d}\theta \int_0^{\frac{9\sin\theta}{\cos^2\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r. \end{aligned}$$

LEBANESE UNIVERSITY Date: 25 Feb. 2009

Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 81

Exercise 1 **[12 pts]**

1° Give the f.e. of order 3 of $\sqrt{1+\sin x}$ near 0.

2° Let f be the function defined over $\mathbb{R}$ by $f(x) = e^{\sqrt{1+\sin x}}$.

(a) Give the f.e. of f to order 3, in neighborhood of $x = 0$.

(b) Find $f'(0)$.

(c) Determine the equation of the tangent (T) to the representative curve (C) of f, at the point of abscissa $x = 0$, and precise the relative position of (C) and (T) in neighborhood of 0.

Exercise 2 **[8 pts]**

Find the equation of the oblique asymptote (T) at $+\infty$, to the representative curve (C) of the function

$$x \longmapsto f(x) = e^{\frac{1}{x}} \sqrt{x(x+2)},$$

and study the position of (C) with respect to (T) at $+\infty$.

Exercise 3 **[12 pts]**

Solve the following differential equations :

1° $y' = 2\dfrac{y}{x} + 1$, with $y(1) = 0$.

2° $y'' - 4y' + 4y = (x+1)e^{-x}$.

Exercise 4 **[14 pts]**

Calculate the following integrals :

1° $I(x) = \displaystyle\int \frac{\mathrm{d}x}{\sqrt{-4x^2+4x+3}}$.

2° $J(x) = \displaystyle\int \frac{\mathrm{d}x}{(1+\tan\frac{x}{2})\sin x}$.

3° $K(x) = \displaystyle\int x\frac{\sin x}{\cos^3 x}\,\mathrm{d}x$.

Exercise 5 ——————————————————— **[10 pts]**

Let

$$I = \int_{-1}^{1} \mathrm{d}x \int_{\sqrt{4-x^2}}^{3} f(x,y)\,\mathrm{d}y.$$

1° Sketch the domain of integration of I.

2° Rewrite the expression of I by reversing the order of integration.

Exercise 6 ——————————————————— **[14 pts]**

Let

$$I = \iint_D \frac{x}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,,$$

where D is the plane region bounded by the curves whose equations are : $y = \sqrt{x}$, $y = 0$ and $x = 3$.

1° Sketch D.

2° Using polar coordinates, calculate I.

Exercise 1

1° Near 0, and up to order 3, we have $\sin x = x - \frac{x^3}{3!} + x^3\varepsilon(x)$. Thus,

$$\begin{aligned}\sqrt{1+\sin x} &= (1+u)^{\frac{1}{2}} + x^3\varepsilon(x)\\ &= 1 + \frac{1}{2}u + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}u^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}u^3 + x^3\varepsilon(x)\end{aligned}$$

with ▷ $u = x - \frac{x^3}{6}$;

▷ $u^2 = \left(x - \frac{x^3}{6}\right)^2 = x^2 + x^3\varepsilon(x)$;

▷ $u^3 = \left(x - \frac{x^3}{6}\right)^3 = x^3 + x^3\varepsilon(x)$.

Which finally gives : $\sqrt{1+\sin x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + x^3\varepsilon(x)$.

2° (a) In neighborhood of 0, and to order 3, we have

$$f(x) = e \times e^{\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48}} + x^3\varepsilon(x) = e\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!}\right) + x^3\varepsilon(x)$$

with ▷ $u = \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48}$;

▷ $u^2 = \left(\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48}\right)^2 = \frac{x^2}{4} - \frac{x^3}{8} + x^3\varepsilon(x)$;

▷ $u^3 = \left(\frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48}\right)^3 = \frac{x^3}{8} + x^3\varepsilon(x)$.

Then, $f(x) = e\left(1 + \frac{x}{2} - \frac{x^3}{16}\right) + x^3\varepsilon(x)$ with $\lim_{x\to 0}\varepsilon(x) = 0$.

(b) From the the f.e. of $f(x)$ in neighborhood of 0, $f'(0) = \frac{e}{2}$.

(c) $y = e + \frac{e}{2}x$ is the equation of the tangent to the representative curve of f at $x = 0$. Moreover, since $f(x) - y \simeq -\frac{e}{16}x^3$ near 0, then we conclude that :

- ► (C) is below (T) if $x \to 0_+$;
- ► (C) is above (T) if $x \to 0_-$.

Exercise 2

Put $x = \dfrac{1}{t}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. Then,

$$f\left(\frac{1}{t}\right) = e^t \times \frac{\sqrt{1+2t}}{t} \Longrightarrow tf\left(\frac{1}{t}\right) = e^t\sqrt{1+2t}.$$

In neighborhood of 0, and up to order 2,

$$tf\left(\frac{1}{t}\right) = \left(1+t+\frac{t^2}{2}\right) \times \left(1+t-\frac{t^2}{2}\right) + t^2\varepsilon(t) = 1+2t+t^2+t^2\varepsilon(t).$$

Thus, $f(x) = x+2+\dfrac{1}{x}+\dfrac{1}{x}\varepsilon(x)$ with $\lim\limits_{x\to+\infty}\varepsilon(x) = 0$.
Then $y = x+2$ is the equation of the asymptote to the infinite branch $(+\infty)$ of the representative curve of f. In addition, as $x \to +\infty$, $f(x) - y \simeq \dfrac{1}{x} > 0$ and the curve is above its asymptote.

Exercise 3

1° The differential equation $(E_1) : y' = 2\dfrac{y}{x} + 1$ can be considered as a linear differential equation of first order (or as a homogeneous one.) Seeking the general solution of (E_1) in the form : $y = u(x) \times v(x)$. Then, $y' = u'v + v'u$. Hence,

$$y' = 2\frac{y}{x}+1 \Rightarrow u'v+v'u = 2\frac{uv}{x}+1 \quad \Rightarrow \quad u'v + u\left[v' - 2\frac{v}{x}\right] = 1.$$

Choosing v such that $v' - 2\dfrac{v}{x} = 0$, set $\dfrac{v'}{v} = \dfrac{2}{x}$. We find, by integration, $\ln|v| = 2\ln|x| + \text{cnst}$. Then we may set $v(x) = x^2$.
For the chosen v, we have $u'v = 1$, so $u' = \dfrac{1}{x^2}$, hence $u(x) = -\dfrac{1}{x} + k$ where $k \in \mathbb{R}$. The general solution of (E_1) is : $y(x) = uv = -x + kx^2$. On the other hand, $y = 0$ for $x = 1$, then, $0 = -1 + k$ so $k = 1$. Thus the solution of (E_1) verifying $y(1) = 0$ is $y = -x + x^2$.
Remark : We may seek the general solution of (E_1) by setting $z = \dfrac{y}{x}$. Then, $y = xz$, hence $y' = xz' + z$. The equation (E_1) becomes

$$xz' = z+1 \quad \Rightarrow \quad \frac{z'}{1+z} = \frac{1}{x}.$$

By integration, we obtain $\ln|1+z| = \ln|x| + k_1$, which is equivalent to $1 + \dfrac{y}{x} = k_2 x$ where k_2 is a real constant. The condition $y(1) = 0$ permits finding $k_2 = 1$, and consequently, $y = -x + x^2$.

2° The solution of $(E_2) : y'' - 4y' + 4y = (x+1)e^{-x}$ is of the form $y = y_H + y_P$ where, y_H is the general solution of the homogeneous equation, and y_P is a particular solution of the complete equation.

Evaluation of y_H The characteristic equation is $r^2 - 4r + 4 = 0$. Thus, $y_H = (c_1 + c_2 x)\, e^{2x}$.

Evaluation of y_P Seeking y_P in the form $y_P = (ax+b)e^{-x}$. Thus, $y' = (-ax + a - b)e^{-x}$ and $y'' = (ax - 2a + b)e^{-x}$. By substitution, it is found that

$$(ax-2a+b)e^{-x} - 4(-ax+a-b)e^{-x} + 4(ax+b)e^{-x} = (x+1)e^{-x}, \quad \forall x \in \mathbb{R},$$

so

$$9ax - 6a + 9b \equiv x + 1.$$

And by identification $a = \dfrac{1}{9}$ and $b = \dfrac{5}{27}$.

The general solution is then, $y(x) = (c_1 + c_2 x)\, e^{2x} + \dfrac{1}{9}\left(x + \dfrac{5}{3}\right) e^{-x}$.

Exercise 4

1° $I(x) = \displaystyle\int \frac{\mathrm{d}x}{\sqrt{4-(2x-1)^2}} = \frac{1}{2}\int \frac{\mathrm{d}(2x-1)}{\sqrt{4-(2x-1)^2}} = \frac{1}{2}\arcsin\frac{2x-1}{2} + k,$ where $k \in \mathbb{R}$.

2° Set $t = \tan\dfrac{x}{2}$, it gives $\mathrm{d}t = \dfrac{1}{2}(1+t^2)\,\mathrm{d}x$. Then

$$J = \int \frac{\dfrac{2\,\mathrm{d}t}{1+t^2}}{(1+t)\dfrac{2t}{1+t^2}} = \int \frac{1}{t(1+t)}\,\mathrm{d}t = \int \left(\frac{1}{t} - \frac{1}{1+t}\right)\mathrm{d}t.$$

It implies that $J(x) = \ln\left|\dfrac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right| + k$ where $k \in \mathbb{R}$.

3° In order to integrate by parts, we set

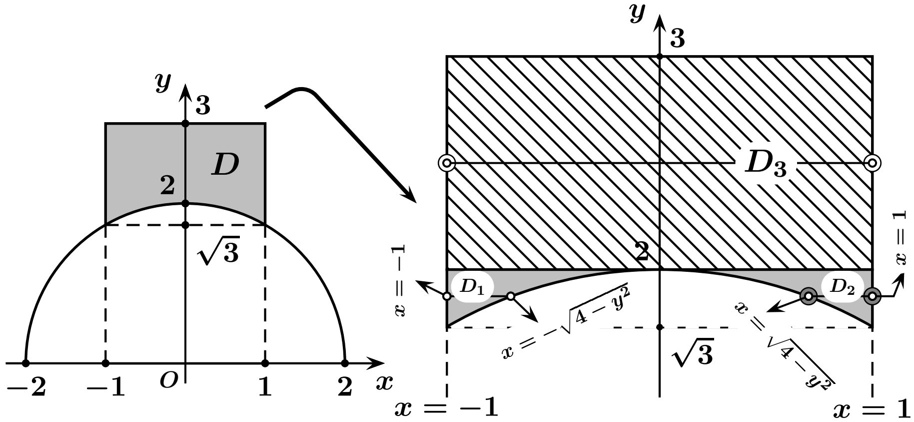
$$\begin{cases} u = x \\ \mathrm{d}v = \dfrac{\sin x}{\cos^3 x}\,\mathrm{d}x \end{cases} \Rightarrow \begin{cases} \mathrm{d}u = \mathrm{d}x; \\ v = \dfrac{1}{2\cos^2 x}. \end{cases}$$

Then we get

$$\begin{aligned} K(x) &= uv - \int v\,\mathrm{d}u = \frac{x}{2\cos^2 x} - \frac{1}{2}\int \frac{\mathrm{d}x}{\cos^2 x} \\ &= \frac{x}{2\cos^2 x} - \frac{1}{2}\tan x + k, \qquad (k \in \mathbb{R}). \end{aligned}$$

Exercise 5

1° The domain D is represented by the following figure.



2° We can write $D = D_1 \cup D_2 \cup D_3$, where the sub-domains D_1, D_2 and D_3 are defined by

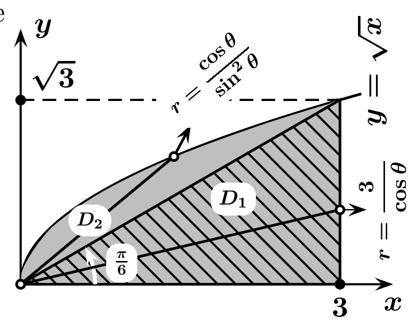
$$D_1 : \begin{cases} \sqrt{3} \le y \le 2 \\ -1 \le x \le -\sqrt{4-y^2} \end{cases} \qquad D_2 : \begin{cases} \sqrt{3} \le y \le 2 \\ \sqrt{4-y^2} \le x \le 1 \end{cases} \qquad D_3 : \begin{cases} 2 \le y \le 3 \\ -1 \le x \le 1 \end{cases}$$

By reversing the order of integration, I is rewritten as

$$\boldsymbol{I = \int_{\sqrt{3}}^{2} \mathrm{d}y \int_{-1}^{-\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{\sqrt{3}}^{2} \mathrm{d}y \int_{\sqrt{4-y^2}}^{1} f(x,y)\,\mathrm{d}x + \int_{2}^{3} \mathrm{d}y \int_{-1}^{1} f(x,y)\,\mathrm{d}x.}$$

Exercise 6

1° The domain D is represented by the adjacent figure : ($D = D_1 \cup D_2$)



2° $D = D_1 \cup D_2$ where the sub-domains D_1 and D_2 are defined by means of polar coordinates as

$$D_1 : \begin{cases} 0 \le \theta \le \dfrac{\pi}{6} \\ 0 \le r \le \dfrac{3}{\cos\theta} \end{cases} \qquad D_2 : \begin{cases} \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le \dfrac{\cos\theta}{\sin^2\theta}. \end{cases}$$

Then integral I is calculated

$$\boldsymbol{\begin{aligned} I &= \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \int_0^{\frac{3}{\cos\theta}} \cos\theta\,\mathrm{d}r + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\cos\theta}{\sin^2\theta}} \cos\theta\,\mathrm{d}r \\ &= 3\int_0^{\frac{\pi}{6}} \mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta\,\mathrm{d}\theta = \frac{\pi}{2} + \Big[-\cot\theta - \theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\pi}{6} + \sqrt{3}. \end{aligned}}$$

LEBANESE UNIVERSITY Date: 18 February 2010
Faculty of Sciences (I)

Final Exam **Math171** **Duration: 2 hours**

ANSWER PAGE 87

Exercise 1 **[13 pts]**

1° Give the finite expansions of order n, in neighborhood of 0, of the following functions :

$$\sqrt{1+t}, \quad n=2; \qquad \ln(1+t), \quad n=4; \qquad \arcsin t, \quad n=2.$$

2° Let $f(x) = \dfrac{\arcsin(1-\sqrt{1+x^2}) - e^{-1}x^2(1+x)^{\frac{1}{x}}}{x-\ln(1+x)}$.

(a) Find the equation of the tangent (T) to the curve (C), representing the extension function of f at the point of abscissa $x=0$.

(b) Determine the relative positions of the tangent (T) and the curve (C) in neighborhood of $x=0$.

Exercise 2 **[8 pts]**

Let $f(x) = (x^2+x+a)\arctan\left(\dfrac{1}{x}\right)$ where a is a real parameter.

1° Give the finite expansion of order 4 of $\arctan(t)$ in neighborhood of 0.

2° Determine the equation of the oblique asymptote (T) to the representative curve (C) of f, at $+\infty$

3° Study, according to the values of a, the relative positions of (T) and (C) at $+\infty$.

Exercise 3 **[10 pts]**

Solve the following differential equations :

1° $y' - y\tan x = \dfrac{1}{\cos^2 x}$.

2° $y'' - 6y' + 9y = (6x-2)e^{3x}$.

Exercise 4 ________________________________ **[12 pts]**

Calculate the following integrals :

1° $\displaystyle\int \cos x \arctan(\sin x)\,\mathrm{d}x$.

2° $\displaystyle\int \frac{\mathrm{d}t}{(1+t^2)t^2}$. Then deduce, $\displaystyle\int \frac{\mathrm{d}x}{\cosh x \sinh^2 x}$.

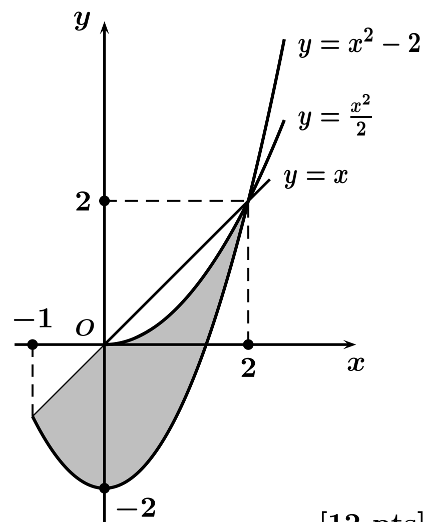
Exercise 5 ________________________________ **[15 pts]**

1° Evaluate $I = \displaystyle\int_0^1 \mathrm{d}y \int_y^1 y e^{x^3}\,\mathrm{d}x$.

2° Let $I = \displaystyle\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where D is the domain shown in the adjacent figure, and defined by :

$$y \geq x^2 - 2, \quad y \leq \frac{x^2}{2}, \quad y \leq x.$$

Using cartesian coordinates, express I in two different forms, each as an iterated succession of two simple integrals.



Exercise 6 ________________________________ **[12 pts]**

Given D the domain of $\mathbb{R}^2$, defined by :

$$y^2 \geq x, \quad x^2 + y^2 - 2y \leq 0 \quad \text{and} \quad x^2 + y^2 + 2x \geq 0.$$

1° Sketch D.

2° Using polar coordinates, calculate the area of D.

Exercise 1

1° ▷ $\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + t^2\varepsilon(t)$;

▷ $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + t^4\varepsilon(t)$;

▷ $\arcsin(t) = t + t^2\varepsilon(t)$.

2° The first term of the f.e. of the denominator $x - \ln(1+x)$ of $f(x)$ is $\frac{x^2}{2}$. A simplification by x^2, should be performed.
To expand $f(x)$ to order 2, it is necessary to give the f.e. of each usual function to order 4 ;

▷ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4\varepsilon(x)$;

▷ To order 2 : $(1+x)^{\frac{1}{x}} = e^{\frac{\ln(1+x)}{x}} = e^{1-\frac{x}{2}+\frac{x^2}{3}} + x^2\varepsilon(x)$

$$= e \times \left(1 - \frac{x}{2} + \frac{11x^2}{24}\right) + x^2\varepsilon(x)$$

▷ For expanding $g(x) = \arcsin(1 - \sqrt{1+x^2})$ to order 4, we expand $g'(x)$ to order 3. Since,

$$g'(x) = -\frac{x}{\sqrt{1+x^2}\sqrt{2\sqrt{1+x^2}-1-x^2}}.$$

It suffices to give the f.e. of $\frac{1}{\sqrt{1+x^2}\sqrt{2\sqrt{1+x^2}-1-x^2}}$ to order 2 ;

▷ $\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + x^2\varepsilon(x)$

▷ $\frac{1}{\sqrt{2\sqrt{1+x^2}-1-x^2}} = \frac{1}{\sqrt{2(1+x^2/2)-1-x^2}} + x^2\varepsilon(x)$

$$= 1 + x^2\varepsilon(x).$$

Thus, $g'(x) = -x\left(1 - \frac{x^2}{2}\right) + x^3\varepsilon(x) = -x + \frac{x^3}{2} + x^3\varepsilon(x)$. And by integration we get

$$g(x) = g(0) - \frac{x^2}{2} + \frac{x^4}{8} + x^4\varepsilon(x) = -\frac{x^2}{2} + \frac{x^4}{8} + x^4\varepsilon(x).$$

Then,

$$f(x) = \frac{-\frac{3}{2} + \frac{x}{2} - \frac{x^2}{3}}{\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4}} + x^2\varepsilon(x) = -3 - x + \frac{x^2}{6} + x^2\varepsilon(x).$$

(a) The equation of (T) is : $y = -3 - x$.

(b) $f(x) - y \underset{0}{\simeq} \dfrac{x^2}{6} > 0$, hence (C) is above (T) in neighborhood 0.

Exercise 2

1° $\arctan(t) = t - \dfrac{t^3}{3} + t^4\varepsilon(t)$.

2° Set $x = \dfrac{1}{t}$ ($x \to +\infty \Leftrightarrow t \to 0_+$). In neighborhood of $t = 0$, and up to order 3, we have

$$\begin{aligned} tf\left(\frac{1}{t}\right) &= \frac{1}{t}(1 + t + at^2)\arctan(t) \\ &= (1 + t + at^2)\left(1 - \frac{t^2}{3}\right) + t^3\varepsilon(t) \\ &= 1 + t + \left(a - \frac{1}{3}\right)t^2 - \frac{t^3}{3} + t^3\varepsilon(t). \end{aligned}$$

It follows that,

$$f(x) = x + 1 + \left(a - \frac{1}{3}\right)\frac{1}{x} - \frac{1}{3x^2} + \frac{1}{x^2}\varepsilon(x).$$

The equation of the asymptote to the curve of f, at $+\infty$, is : $y = x+1$.

3° ▷ If $a > \dfrac{1}{3}$; $f(x) - y \underset{+\infty}{\simeq} \left(a - \dfrac{1}{3}\right)\dfrac{1}{x} > 0$, then (C) is above the asymptote.

▷ If $a < \dfrac{1}{3}$; $f(x) - y \underset{+\infty}{\simeq} \left(a - \dfrac{1}{3}\right)\dfrac{1}{x} < 0$, then (C) is below the asymptote.

▷ If $a = \dfrac{1}{3}$; $f(x) - y \underset{+\infty}{\simeq} -\dfrac{1}{3x^2} < 0$, then (C) is below the asymptote.

Exercise 3

1° $y' - y\tan x = \dfrac{1}{\cos^2 x}$ is a linear differential equation of first order. Seeking y in the form $y = uv$. The considered differential equation is written as

$$u'v + \left[v' - v\frac{\sin x}{\cos x}\right] u = \frac{1}{\cos^2 x}.$$

We choose the function v such that

$$v' - v\frac{\sin x}{\cos x} = 0 \quad \text{so,} \quad \frac{\mathrm{d}v}{v} = \frac{\sin x}{\cos x}\,\mathrm{d}x \quad \text{hence} \quad v = \frac{1}{\cos x}.$$

Since $u'v = \dfrac{1}{\cos^2 x}$ thus, $u' = \dfrac{1}{\cos x}$. Therefore,

$$u(x) = \int \frac{\mathrm{d}x}{\cos x} = \int \frac{\mathrm{d}(\sin x)}{1 - \sin^2 x} = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + k,$$

where $k \in \mathbb{R}$. It follows that $y(x) = \dfrac{1}{2\cos x}\ln\dfrac{1+\sin x}{1-\sin x} + \dfrac{k}{\cos x}$.

2° We seek the general solution $y(x)$ of $y'' - 6y' + 9y = (6x-2)e^{3x}$ in the form $y(x) = y_H(x) + y_p(x)$, where

▷ y_H is the general solution of the homogeneous equation

$$y'' - 6y' + 9y = 0.$$

▷ y_p is a particular solution of

$$y'' - 6y' + 9y = (6x-2)e^{3x}.$$

- **Evaluation of y_H :** The associated characteristic equation is

$$r^2 - 6r + 9 = 0 \qquad (C)$$

which admits a double roots $r_1 = r_2 = 3$. Then, $y_H = (c_1 + c_2 x)\,e^{3x}$.

- **Evaluation of y_p :** $f(x) = (6x-2)e^{3x} = P_1(x)e^{\alpha x}$. Since $\alpha = 3$ is a double root of (C), we seek a particular solution y_p in the form $y_p(x) = x^2(a+bx)e^{3x}$. It is found that $a = -1$ and $b = 1$.

The general solution of the considered differential equation is

$$y(x) = y_H(x) + y_p(x) = (c_1 + c_2 x)\,e^{3x} + (x-1)x^2 e^{3x}.$$

Exercise 4

1° **Calculation of $I = \int \cos x \arctan(\sin x)\,\mathrm{d}x$.** Integrate I by parts by setting :

$$u = \arctan(\sin x) \quad \text{and} \quad \mathrm{d}v = \cos x\,\mathrm{d}x$$

then,

$$\mathrm{d}u = \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x \quad \text{and} \quad v = \sin x.$$

Then we obtain, $I = \sin x \arctan(\sin x) - \int \frac{\sin x \cos x\,\mathrm{d}x}{1+\sin^2 x}$.

Calculation of $J = \int \frac{\sin x \cos x\,\mathrm{d}x}{1+\sin^2 x}$.

▷ First idea : Set $t = \sin x$. Thus,

$$J = \int \frac{t\,\mathrm{d}t}{1+t^2} = \frac{1}{2}\ln(1+t^2) + \text{cnst} = \frac{1}{2}\ln(1+\sin^2 x) + \text{cnst}.$$

▷ Second idea : $J = \int \frac{\sin(2x)\,\mathrm{d}x}{3-\cos(2x)} = \frac{1}{2}\ln(3-\cos(2x)) + \text{cnst}.$

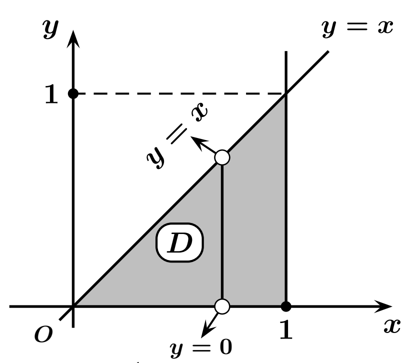
2° $I = \int \frac{1}{t^2(1+t^2)}\,\mathrm{d}t = \int \left(\frac{1}{t^2} - \frac{1}{1+t^2}\right)\mathrm{d}t = -\frac{1}{t} - \arctan t +$ cnst.

$$\begin{aligned} J(x) &= \int \frac{\cosh x\,\mathrm{d}x}{\cosh^2 x \sinh^2 x}\,\mathrm{d}x \underset{t=\sinh x}{=} \int \frac{\mathrm{d}t}{t^2(1+t^2)} \\ &= I(t) = I(\sinh x) = -\frac{1}{\sinh x} - \arctan(\sinh x) + \text{cnst}. \end{aligned}$$

Exercise 5

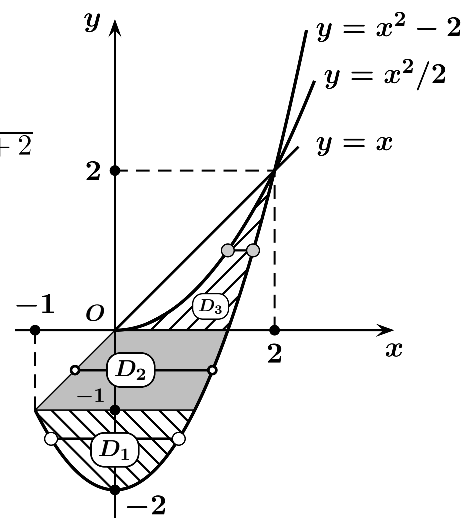
1° The change of order of integration enables the calculation of the double integral $I = \int_0^1 \mathrm{d}y \int_y^1 y e^{x^3}\,\mathrm{d}x$. Holding x fixed (vertically), D is defined by :

$$D : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$



Thus, $I = \int_0^1 e^{x^3}\,\mathrm{d}x \int_0^x y\,\mathrm{d}y = \frac{1}{2}\int_0^1 x^2 e^{x^3}\,\mathrm{d}x = \frac{1}{6}(e-1).$

2° $D = D_1 \cup D_2 \cup D_3$, where



▷ $D_1 : \begin{cases} -2 \le y \le -1 \\ -\sqrt{y+2} \le x \le \sqrt{y+2} \end{cases}$

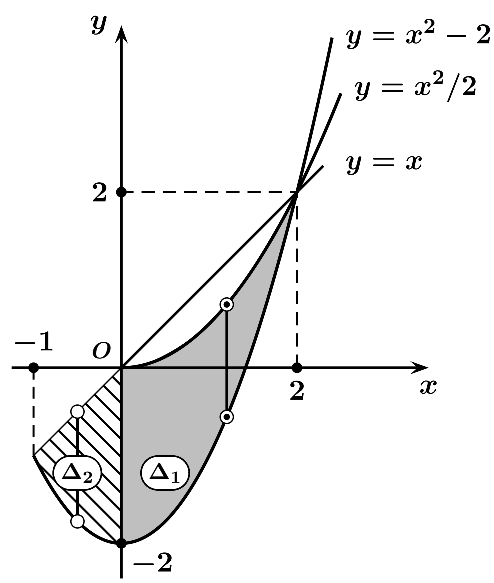
▷ $D_2 : \begin{cases} -1 \le y \le 0 \\ y \le x \le \sqrt{y+2} \end{cases}$

▷ $D_3 : \begin{cases} 0 \le y \le 2 \\ \sqrt{2y} \le x \le \sqrt{y+2} \end{cases}$

Thus,

$$I = \int_{-2}^{-1} \mathrm{d}y \int_{-\sqrt{y+2}}^{\sqrt{y+2}} f(x,y)\,\mathrm{d}x + \int_{-1}^{0} \mathrm{d}y \int_{y}^{\sqrt{y+2}} f(x,y)\,\mathrm{d}x + \int_{0}^{2} \mathrm{d}y \int_{\sqrt{2y}}^{\sqrt{y+2}} f(x,y)\,\mathrm{d}x.$$

Or equivalently,



$D = \Delta_1 \cup \Delta_2$, where

▷ $\Delta_1 : \begin{cases} -1 \le x \le 0 \\ x^2 - 2 \le y \le x \end{cases}$

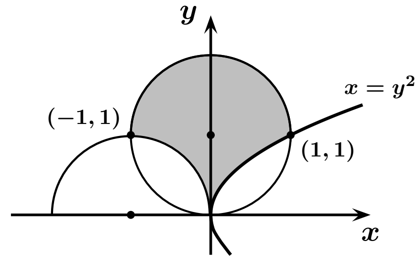
▷ $\Delta_2 : \begin{cases} 0 \le x \le 2 \\ x^2 - 2 \le y \le \dfrac{x^2}{2} \end{cases}$

Thus,

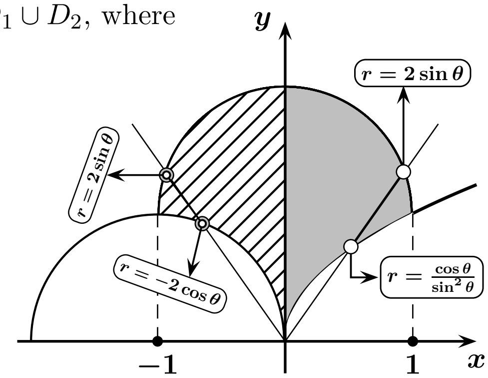
$$I = \int_{-1}^{0} \mathrm{d}x \int_{x^2-2}^{x} f(x,y)\,\mathrm{d}y + \int_{0}^{2} \mathrm{d}x \int_{x^2-2}^{\frac{x^2}{2}} f(x,y)\,\mathrm{d}y.$$

Exercise 6

1° Graphical representation of the domain D ;



2° In polar coordinates $D = D_1 \cup D_2$, where



▷ $D_1 : \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2} \\ \dfrac{\cos\theta}{\sin^2\theta} \le r \le 2\sin\theta \end{cases}$

▷ $D_2 : \begin{cases} \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{4} \\ -2\cos\theta \le r \le 2\sin\theta \end{cases}$

$$\begin{aligned}\textbf{Area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y = \iint_{D_1\cup D_2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \iint_{D_1} r\,\mathrm{d}r\,\mathrm{d}\theta + \iint_{D_2} r\,\mathrm{d}r\,\mathrm{d}\theta = I_1 + I_2.\end{aligned}$$

► $$\begin{aligned} I_1 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{\cos\theta}{\sin^2\theta}}^{2\sin\theta} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(4\sin^2\theta - \frac{\cos^2\theta}{\sin^4\theta}\right)\mathrm{d}\theta \\ &= \frac{1}{2}\left[2\theta - \sin(2\theta) + \frac{1}{3}\cot^3\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{3}.\end{aligned}$$

► $$\begin{aligned} I_2 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \mathrm{d}\theta \int_{-2\cos\theta}^{2\sin\theta} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 4\left(\sin^2\theta - \cos^2\theta\right)\mathrm{d}\theta \\ &= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 2\cos(2\theta)\,\mathrm{d}\theta = -\Big[\sin(2\theta)\Big]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = 1.\end{aligned}$$

Thus, $\textbf{Area}(D) = \dfrac{\pi}{4} + \dfrac{4}{3}$.

LEBANESE UNIVERSITY Date: 17 February 2011
Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 96

Exercise 1 **[9 pts]**

Let α be a real parameter, and consider

$$f(x) = \sqrt[3]{\alpha x + x^2 + x^3}.$$

1° Set $x = \dfrac{1}{t}$. Find the finite expansion of order 3 of $tf\left(\dfrac{1}{t}\right)$ in neighborhood of $t = 0$.

2° Deduce the equation of the asymptote (D) to the representative curve (C) of f, at $+\infty$.

3° Discuss according to the values of α the relative positions of (C) and (D), at $+\infty$.

Exercise 2 **[12 pts]**

1° Give the finite expansions of order 2 in neighborhood of $x = 0$, of the following functions :

$$f(x) = \cos\left(\frac{x\sqrt{2}}{e}\right), \qquad g(x) = \sin(ax), \quad a \in \mathbb{R};$$

$$h(x) = \ln\left(e + 2x + \frac{2x^2}{e}\right), \qquad m(x) = \sqrt{3 + h(x)}.$$

2° Give the finite expansions near 0, to order 2, of $\dfrac{\arctan t}{t}$. Deduce the f.e. of $n(x) = \dfrac{e}{x}\arctan\dfrac{x}{e}$ near 0, to order 2.

3° Determine a such that the function

$$F(x) = \frac{m(x) - 3n(x) + f(x) - g(x)}{x^2}$$

admits a finite limit as x tends to 0. What is then this limit ?

Exercise 3 **[8 pts]**

1° Calculate $I = \displaystyle\int \frac{x+4}{x^2+2x+5}\,\mathrm{d}x$.

2° (a) Determine the real constants A, B and C such that

$$\frac{1}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}.$$

(b) Calculate

$$J = \int \frac{1}{(1+t^2)(1+t)}\,\mathrm{d}t, \quad \text{and} \quad K = \int \frac{\sin x\,\mathrm{d}x}{(1+\cos^2 x)(1+\cos x)}.$$

(c) Calculate $L = \displaystyle\int \arctan(\cos x)\frac{\sin x}{(1+\cos x)^2}\,\mathrm{d}x$.

Exercise 4 **[8 pts]**

1° Calculate

$$\int \frac{\mathrm{d}t}{t(1+t)} \quad \text{and} \quad \int \frac{\tan x}{1+\cos x}\,\mathrm{d}x.$$

2° Determine the solution of

$$y' + \frac{\sin x}{1+\cos x}y = \tan x$$

that satisfies $y(0) = 2\ln 2$.

Exercise 5 **[7 pts]**

Solve the following differential equation : $y'' + 3y' = 3 + e^{-3x}$.

Exercise 6 **[13 pts]**

Consider the shaded domain D of figure 1, defined by :

$$D = \left\{(x,y) \in \mathbb{R}^2 \; ; \; 0 \le y \le 1, \; y^2 \le x \le \sqrt{4-y^2}\right\}.$$

Let f be an integrable function of two variables x and y, over D.

1° Using polar coordinates, write down the expression of the double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

2° Using polar coordinates, calculate the area of D.

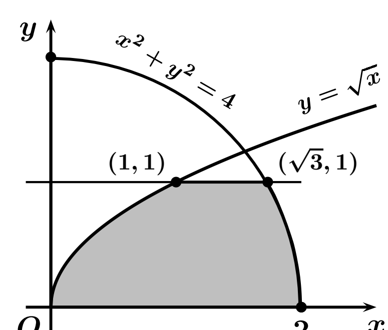


Figure 1.

Exercise 7 —————————————————————— **[13 pts]**

Choose and then treat one of the two following questions :

<u>**Question 1 :**</u>

Consider the domain D of $\mathbb{R}^2$, shaded in figure 2, and defined by :

$$D = \begin{cases} y \leq 1, \\ x^2 + y^2 \leq 2, \\ y \geq -\dfrac{x^2}{4} + \dfrac{1}{2}. \end{cases}$$

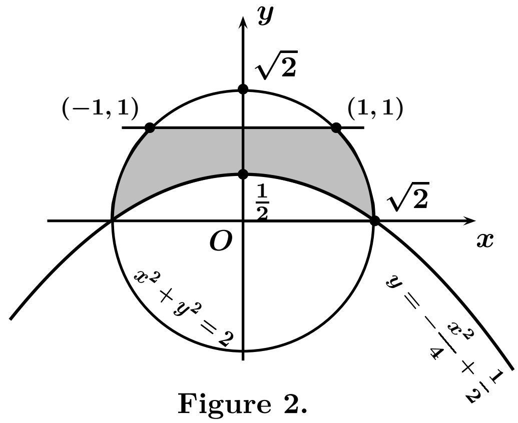


Figure 2.

Let f be an integrable function of two variables x and y, over D.

Using cartesian coordinates, write down the double integral

$$I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

in two different forms, each as an iterated succession of two simple integrals.

<u>**Question 2 :**</u>

Consider the shaded domain D of figure 3, defined by :

$$D = \{(x, y) \in \mathbb{R}^2 \; ; \; y \geq 0, \; -2 \leq x \leq 2, \; x^2 + y^2 \leq 8, \; x^2 + y^2 \geq 2x, \; x^2 + y^2 \geq -2x\}.$$

Let f be a function of two variables x and y, that is integrable on D.

Using cartesian coordinates, write down the following double integral :

$$I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

in two different forms, each as an iterated succession of two simple integrals.

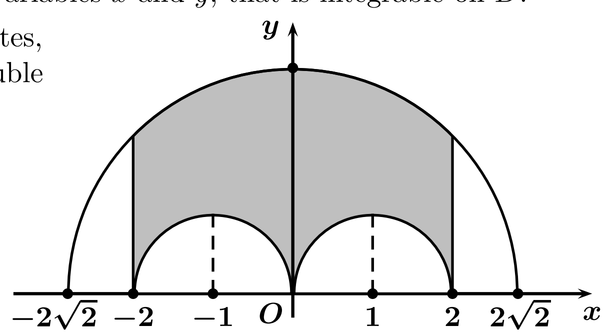


Figure 3.

Exercise 1

1° First of all we have

$$tf\left(\frac{1}{t}\right) = \sqrt[3]{1+t+\alpha t^2} = \left(1+t+\alpha t^2\right)^{\frac{1}{3}}.$$

Expanding in neighborhood of $t=0$ to order 3,

$$\begin{aligned} tf\left(\frac{1}{t}\right) &= (1+u)^{\frac{1}{3}} \\ &= 1+\frac{1}{3}u+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}u^2+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}u^3+t^3\varepsilon(t) \end{aligned}$$

with,

- ▷ $u = t+\alpha t^2$;
- ▷ $u^2 = (t+\alpha t^2)^2 = t^2+2\alpha t^3+t^3\varepsilon(t)$;
- ▷ $u^3 = (t+\alpha t^2)^3 = t^3+t^3\varepsilon(t)$.

Hence, $tf\left(\frac{1}{t}\right) = 1+\frac{t}{3}+\frac{1}{9}(3\alpha-1)t^2+\frac{1}{81}(5-18\alpha)t^3+t^3\varepsilon(t)$.

2° At $+\infty$, we have

$$f(x) = x+\frac{1}{3}+(3\alpha-1)\frac{1}{9x}+(5-18\alpha)\frac{1}{81x^2}+\frac{1}{x^2}\varepsilon(x)$$

with, $\lim\limits_{x\to+\infty}\varepsilon(x)=0$.

Then, the equation of the asymptote (D) is : $y = x+\frac{1}{3}$.

3° ▷ When $\alpha \neq \frac{1}{3}$; $f(x)-y \underset{+\infty}{\simeq} (3\alpha-1)\frac{1}{9x}$. Then,

- ► (C) is above (D) for $\alpha > \frac{1}{3}$;
- ► (C) is below (D) for $\alpha < \frac{1}{3}$.

▷ When $\alpha = \frac{1}{3}$; $f(x)-y \underset{+\infty}{\simeq} -\frac{1}{81x^2} < 0$, then (C) is below its asymptote (D) at $+\infty$.

Exercise 2

1° In neighborhood of $x = 0$, and to order 2,we have

- $f(x) = \cos\left(\dfrac{x\sqrt{2}}{e}\right) = 1 - \dfrac{x^2}{e^2} + x^2\varepsilon(x)$.
- $g(x) = \sin(ax) = ax + x^2\varepsilon(x)$.
- $$\begin{aligned} h(x) &= \ln\left(e + 2x + \frac{2x^2}{e}\right) = \ln e + \ln\left(1 + \frac{2x}{e} + \frac{2x^2}{e^2}\right) \\ &= 1 + \frac{2x}{e} + \frac{2x^2}{e^2} - \frac{1}{2}\left(\frac{2x}{e} + \frac{2x^2}{e^2}\right)^2 + x^2\varepsilon(x) \\ &= 1 + \frac{2x}{e} + x^2\varepsilon(x). \end{aligned}$$
- $$\begin{aligned} m(x) = \sqrt{3 + h(x)} &= \sqrt{4 + \frac{2x}{e}} + x^2\varepsilon(x) = 2\left(1 + \frac{x}{2e}\right)^{\frac{1}{2}} + x^2\varepsilon(x) \\ &= 2\left(1 + \frac{x}{4e} - \frac{x^2}{32e^2}\right) + x^2\varepsilon(x) \\ &= 2 + \frac{x}{2e} - \frac{x^2}{16e^2} + x^2\varepsilon(x). \end{aligned}$$

2° In neighborhood of $t = 0$, and to order 2, we have

$$\arctan'(t) = \frac{1}{1 + t^2} = 1 - t^2 + t^2\varepsilon(t).$$

Integrating,

$$\arctan t = \arctan 0 + t - \frac{t^3}{3} + t^3\varepsilon(t).$$

It follows that

$$\frac{1}{t}\arctan t = 1 - \frac{t^2}{3} + t^2\varepsilon(t).$$

Thus, $n(x) = 1 - \dfrac{x^2}{3e^2} + x^2\varepsilon(x)$.

3° $F(x) = \dfrac{m(x) - 3n(x) + f(x) - g(x)}{x^2} = \left(\dfrac{1}{2e} - a\right)\dfrac{1}{x} - \dfrac{1}{16e^2} + \varepsilon(x)$.

For $a = \dfrac{1}{2e}$, we have $\lim\limits_{x\to 0} F(x) = -\dfrac{1}{16e^2}$.

Exercise 3

1° $$I = \frac{1}{2}\int \frac{2x+8}{x^2+2x+5}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x + 3\int \frac{\mathrm{d}(x+1)}{(x+1)^2+2^2}$$
$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\arctan\frac{x+1}{2} + \text{cnst}.$$

2° (a) $$\frac{1}{(1+t^2)(1+t)} = \frac{1}{2(1+t)} + \frac{1-t}{2(1+t^2)}.$$

(b) $$J = \int \frac{\mathrm{d}t}{(1+t^2)(1+t)} = \frac{1}{2}\left[\ln|1+t| + \arctan t - \frac{1}{2}\ln(1+t^2)\right] + \text{cnst}.$$

With $t = \cos x$, $K = -\int \frac{\mathrm{d}t}{(1+t^2)(1+t)} = -J(t) = -J(\cos x)$.

(c) Integrating by parts with $u = \arctan x$ and $\mathrm{d}v = \frac{\sin x\,\mathrm{d}x}{(1+\cos x)^2}$, we obtain

$$L = uv - \int v\,\mathrm{d}u = \frac{\arctan\cos x}{1+\cos x} + \int \frac{\sin x}{(1+\cos^2 x)(1+\cos x)}\,\mathrm{d}x$$
$$= \frac{\arctan(\cos x)}{1+\cos x} + K(x).$$

Exercise 4

1° $$I(t) = \int \frac{\mathrm{d}t}{t(1+t)} = \ln\left|\frac{t}{1+t}\right| + \text{cnst}.$$

$$J(x) = \int \frac{\tan x\,\mathrm{d}x}{1+\cos x} = \int \frac{-\mathrm{d}(\cos x)}{\cos x(1+\cos x)} = -I(\cos x) = \ln\left|\frac{1+\cos x}{\cos x}\right| + \text{cnst}.$$

2° It is the case of a linear differential equation of first order. Seeking y in the form of $y = uv$. Thus the considered differential equation is rewritten as

$$u'v + u\left[v' + \frac{\sin x}{1+\cos x}v\right] = \tan x.$$

Choosing the function v such that

$$v' + \frac{\sin x}{1+\cos x}v = 0 \Rightarrow \frac{\mathrm{d}v}{v} = -\frac{\sin x}{1+\cos x}\,\mathrm{d}x \Rightarrow v = 1+\cos x.$$

Since $u'v = \tan x$ hence, $u' = \frac{\tan x}{1+\cos x}$. From question 1° :

$$u(x) = \ln\left|\frac{1+\cos x}{\cos x}\right| + c$$

where $c \in \mathbb{R}$. It follows that $y(x) = \left(\ln\left|\frac{1+\cos x}{\cos x}\right| + c\right)(1+\cos x)$. On the other hand $y(0) = 2\ln 2$ gives $c = 0$.

Exercise 5

We seek the general solution $y(x)$ of $y'' + 3y' = 3 + e^{-3x}$ in the form $y(x) = y_H(x) + y_{p_1}(x) + y_{p_2}(x)$, where

▷ y_H is the general solution of the homogeneous equation $y'' + 3y' = 0$;

▷ y_{p_1} is a particular solution of $y'' + 3y' = 3$;

▷ y_{p_2} is a particular solution of $y'' + 3y' = e^{-3x}$.

- **Calculation of y_H :** The associated characteristic equation is written as

$$r^2 + 3r = 0; \qquad (C)$$

which admits two roots $r_1 = 0$ and $r_2 = -3$. Hence, $y_H = c_1 + c_2 e^{-3x}$.

- **Calculation of y_{p_1} :** $f_1(x) = 3 = P_0 e^{\alpha x}$. Since $\alpha = 0$ is a simple root of (C), then $y_{p_1}(x) = ax$. We find $a = 1$.
- **Calculation of y_{p_2} :** $f_2(x) = e^{-3x} = P_0 e^{\alpha x}$. Since $\alpha = -3$ is a simple root of (C), then $y_{p_2}(x) = axe^{-3x}$. We find $a = -\dfrac{1}{3}$.

The general solution of the given differential equation is,

$$y(x) = y_H(x) + y_{p_1}(x) + y_{p_2}(x) = c_1 + c_2 e^{-3x} + x - \frac{1}{3} x e^{-3x}.$$

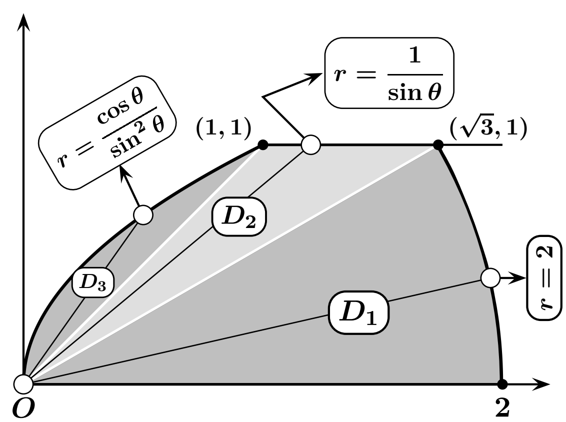
Exercise 6

1° $D = D_1 \cup D_2 \cup D_3$, where

$$D_1 : \begin{cases} 0 \le \theta \le \dfrac{\pi}{6} \\ 0 \le r \le 2 \end{cases}$$

$$D_2 : \begin{cases} \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{4} \\ 0 \le r \le \dfrac{1}{\sin\theta} \end{cases}$$

$$D_3 : \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le \dfrac{\cos\theta}{\sin^2\theta} \end{cases}$$



Thus,

$$I = \int_0^{\frac{\pi}{6}} d\theta \int_0^2 f(r\cos\theta, r\sin\theta) r\, dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta \int_0^{\frac{1}{\sin\theta}} f(r\cos\theta, r\sin\theta) r\, dr$$

$$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{\frac{\cos\theta}{\sin^2\theta}} f(r\cos\theta, r\sin\theta) r\, dr.$$

$$2^\circ \quad \mathbf{Area(D)} = \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \int_0^2 r\,\mathrm{d}r + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{1}{\sin\theta}} r\,\mathrm{d}r + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\cos\theta}{\sin^2\theta}} r\,\mathrm{d}r$$

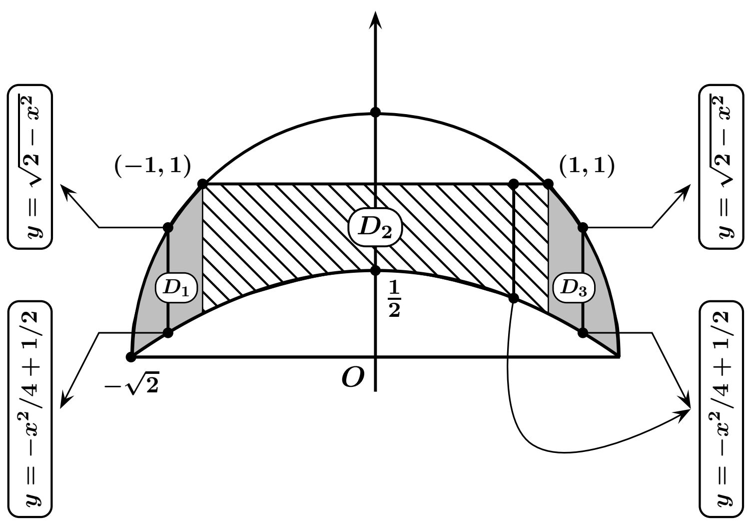
$$= \frac{1}{2}\left(4\int_0^{\frac{\pi}{6}} \mathrm{d}\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2\theta}\,\mathrm{d}\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sin^4\theta}\,\mathrm{d}\theta\right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} - \Big[\cot\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \Big[\frac{1}{3}\cot^3\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{1}{3}.$$

Exercise 7

Question 1 :

► By holding x fixed (vertically) in the first iterative step ;



Therefore,

$$D = D_1 \cup D_2 \cup D_3, \text{ where}$$

$$D_1 : \begin{cases} -\sqrt{2} \le x \le -1 \\ -\dfrac{x^2}{4} + \dfrac{1}{2} \le y \le \sqrt{2 - x^2} \end{cases} \qquad D_2 : \begin{cases} -1 \le x \le 1 \\ -\dfrac{x^2}{4} + \dfrac{1}{2} \le y \le 1 \end{cases}$$

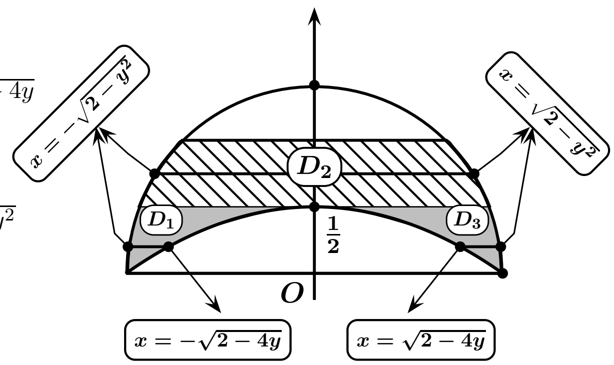
$$\text{and} \quad D_3 : \begin{cases} 1 \le x \le \sqrt{2} \\ -\dfrac{x^2}{4} + \dfrac{1}{2} \le y \le \sqrt{2 - x^2} \end{cases}$$

► Now by holding y fixed (horizontally) in the first iteration ;
$D = D_1 \cup D_2 \cup D_3$, where

$$D_1 : \begin{cases} 0 \leq y \leq \dfrac{1}{2} \\ -\sqrt{2-y^2} \leq x \leq -\sqrt{2-4y} \end{cases}$$

$$D_2 : \begin{cases} \dfrac{1}{2} \leq y \leq 1 \\ -\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2} \end{cases}$$

$$D_3 : \begin{cases} 0 \leq y \leq \dfrac{1}{2} \\ \sqrt{2-4y} \leq x \leq \sqrt{2-y^2} \end{cases}$$



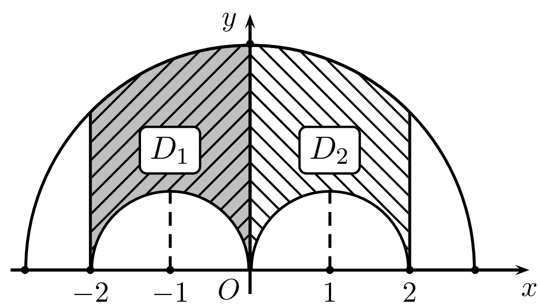
Question 2 :

► First, fix x (vertically) ;

$D = D_1 \cup D_2$, where

$$D_1 : \begin{cases} -2 \leq x \leq 0 \\ \sqrt{-2x-x^2} \leq y \leq \sqrt{8-x^2} \end{cases}$$

$$D_2 : \begin{cases} 0 \leq x \leq 2 \\ \sqrt{2x-x^2} \leq y \leq \sqrt{8-x^2} \end{cases}$$



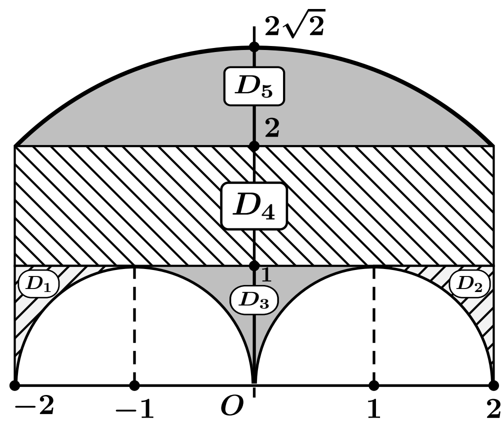
► First, fixing y (horizontally) ;
$D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$,

where

$$D_1 : \begin{cases} 0 \leq y \leq 1 \\ -2 \leq x \leq -1 - \sqrt{1-y^2} \end{cases}$$

$$D_2 : \begin{cases} 0 \leq y \leq 1 \\ 1 + \sqrt{1-y^2} \leq x \leq 2 \end{cases}$$

$$D_3 : \begin{cases} 0 \leq y \leq 1 \\ -1 + \sqrt{1-y^2} \leq x \leq 1 - \sqrt{1} - y^2 \end{cases}$$



$$D_4 : \begin{cases} 1 \leq y \leq 2 \\ -2 \leq x \leq 2 \end{cases} \qquad D_5 : \begin{cases} 2 \leq y \leq 2\sqrt{2} \\ -\sqrt{8-y^2} \leq x \leq \sqrt{8-y^2} \end{cases}$$

Note : Writing down the double integrals, in details, in questions 1 and 2 is left to the student as an exercise.

LEBANESE UNIVERSITY Date: 5 March 2012
Faculty of Sciences (I)

Final Exam **Math171** **Duration: 2 hours**

ANSWER PAGE 105

Exercise 1 **[12 points]**

Let $f(x) = (1+2x)^{\frac{1}{2\sin x}} - \sqrt{\dfrac{1-2x}{1+x}}e^{\frac{1}{1-x}}$.

1° Give the finite expansion of $f(x)$ to order 2 in neighborhood of $x = 0$.

2° Show that $f(x)$ could be extended by continuity to $x = 0$. Let g be its extension.

3° Determine the equation of the tangent (Δ) to the curve (C) of g at 0. Precise the relative position of (Δ) and (C) near $x = 0$.

Exercise 2 **[8 points]**

Let $f(x) = \sqrt[3]{1+x^3} + \sqrt{a+x+x^2}$, where $a \in \mathbb{R}$. Our aim is to determine the asymptote and its position relative to the curve of f at $-\infty$. Set $t = \dfrac{1}{x}$ $(x \to -\infty \iff t \to 0_-)$.

1° Give the finite expansion of $tf\left(\dfrac{1}{t}\right)$ to order 3 in neighborhood of $t = 0$.

2° Determine the equation of the asymptote (Δ) to the curve (C) of f. Precise the position of (Δ) relative to (C) in neighborhood of $-\infty$, according to the values of a.

Exercise 3 **[5 points]**

Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by :

$$f(x) = \frac{1}{1+e^x}, \qquad g(x) = \frac{1}{1+e^{-x}}.$$

Consider $I = \displaystyle\int_0^1 f(x)\,\mathrm{d}x$ and $J = \displaystyle\int_0^1 g(x)\,\mathrm{d}x$.

1° Calculate $f(x) + g(x)$, and deduce $I + J$ (without their calculation).

2° Calculate J and then deduce I.

Exercise 4 ——— **[6 points]**

1° Evaluate $I = \displaystyle\int \frac{\mathrm{d}x}{3 + 5\cos x}$.

2° Evaluate $J = \displaystyle\int (\ln x)^2 \,\mathrm{d}x$.

Exercise 5 ——— **[6 points]**

1° Determine the real numbers a, b and c such that :

$$\frac{1}{(1+t)(1+t^2)} = \frac{a}{1+t} + \frac{bt+c}{1+t^2}.$$

Calculate : $I = \displaystyle\int_0^1 \frac{\mathrm{d}t}{(1+t)(1+t^2)}$.

2° Calculate $J = \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,\mathrm{d}x$, setting $t = \tan x$.

3° Verify that : $\dfrac{1}{1+\tan x} = \dfrac{1}{2}\left[1 + \dfrac{\cos x - \sin x}{\cos x + \sin x}\right]$.

Deduce J.

Exercise 6 ——— **[10 points]**

Consider the shaded domain D of the plane $\mathbb{R}^2$ (see figure 1). Write, in two different forms, the double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

as a succession of two iterated simple integrals by using cartesian coordinates.

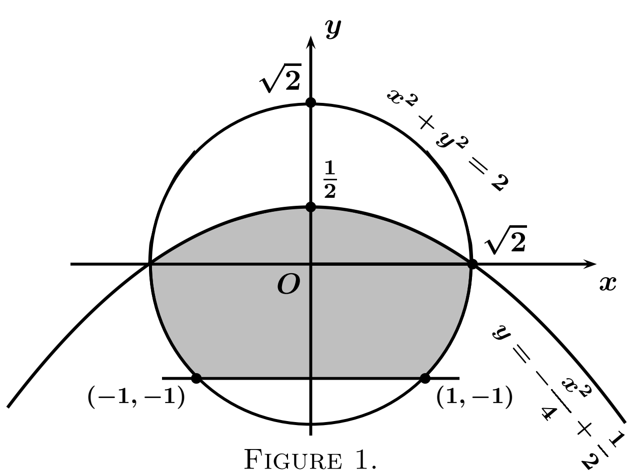


Figure 1.

Exercise 7 **[15 points]**

1° Consider the shaded domain D, shown in figure 2, defined by :

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq y \leq 2, \ \ x^2 + y^2 \leq 8, \ x^2 + y^2 \geq 2x, \ x^2 + y^2 \geq -2x\}.$$

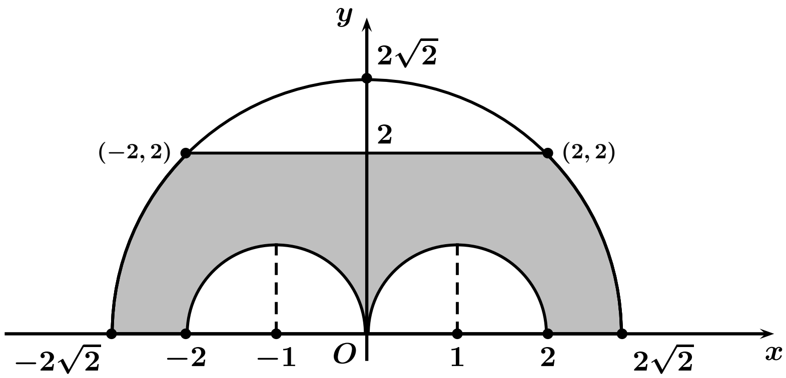


FIGURE 2.

Write, in two different forms, the double integral $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$, as a succession of two iterated simple integrals by using cartesian coordinates.

2° Using polar coordinates, calculate the area of the shaded domain Δ of the plane $\mathbb{R}^2$.

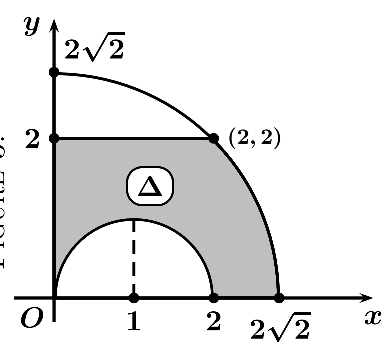


FIGURE 3.

Exercise 8 **[8 points]**

1° (a) Verify that $\dfrac{1}{x(x^2-1)} = -\dfrac{1}{x} + \dfrac{x}{x^2-1}$.

(b) Solve the differential equation : $x(x^2-1)y' + 2y = x^3$.

2° Solve : $xy' + \sqrt{xy}\,y' = y$.

Exercise 1

1° In the neighborhood of $x = 0$ we have :

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + x^3\varepsilon(x)$$

and

$$\frac{\ln(1+2x)}{\sin x} = \frac{1 - x + \dfrac{4x^2}{3}}{1 - \dfrac{x^2}{6}} + x^2\varepsilon(x) = 1 - x + \frac{3}{2}x^2 + x^2\varepsilon(x).$$

Thus

$$\begin{aligned}(1+2x)^{\frac{1}{2\sin x}} &= e^{\frac{\ln(1+2x)}{\sin x}} = e \times e^{-x+\frac{3}{2}x^2} + x^2\varepsilon(x)\\ &= e\left(1 - x + 2x^2\right) + x^2\varepsilon(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sqrt{\frac{1-2x}{1+x}}e^{\frac{1}{1-x}} &= (1-2x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}e^{1+x+x^2} + x^2\varepsilon(x)\\ &= e \times \left(1 - x - \frac{1}{2}x^2\right)\left(1 - \frac{x}{2} + \frac{3}{8}x^2\right)\left(1 + x + \frac{3}{2}x^2\right) + x^2\varepsilon(x)\\ &= e\left(1 - \frac{1}{2}x + \frac{3}{8}x^2\right) + x^2\varepsilon(x).\end{aligned}$$

Therefore,

$$f(x) = e \times \left(-\frac{x}{2} + \frac{13}{8}x^2\right) + x^2\varepsilon(x).$$

2° We have $\lim\limits_{x\to 0} f(x) = 0$. Hence f is extendable by continuity to $x = 0$.

3° $y = -\dfrac{e}{2}x$.

Since $g(x) - y \underset{0}{\simeq} \dfrac{13e}{8}x^2 > 0$ it results that the curve of g is situated above the tangent near $x = 0$.

Exercise 2

1° In neighborhood of $t = 0_-$ and up to order 3 :

$$tf\left(\frac{1}{t}\right) = (1+t^3)^{\frac{1}{3}} - (1+t+at^2)^{\frac{1}{2}} = -\frac{t}{2} + \frac{1}{2}\left(\frac{1}{4} - a\right)t^2 + \frac{1}{4}\left(\frac{13}{12} + a\right)t^3 + t^3\varepsilon(t).$$

2° We have : $f(x) = -\dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{1}{4} - a\right)\dfrac{1}{2x} + \left(\dfrac{13}{12} + a\right)\dfrac{1}{4x^2} + \dfrac{1}{x^2}\varepsilon(x).$

The equation of the asymptote is $y = -\dfrac{1}{2}$.

- If $a = \dfrac{1}{4}$, $f(x) - y \underset{-\infty}{\simeq} \dfrac{1}{3x^2} > 0$, then the curve of f is above the asymptote at $-\infty$.
- If $a < \dfrac{1}{4}$, $f(x) - y \underset{-\infty}{\simeq} \dfrac{1}{2}\left(\dfrac{1}{4} - a\right)\dfrac{1}{2x} < 0$, then the curve of f is below the asymptote at $-\infty$.
- If $a > \dfrac{1}{4}$, $f(x) - y \underset{-\infty}{\simeq} \dfrac{1}{2}\left(\dfrac{1}{4} - a\right)\dfrac{1}{2x} > 0$, then the curve of f is above the asymptote at $-\infty$.

Exercise 3

1° $f(x) + g(x) = \dfrac{1}{1+e^x} + \dfrac{1}{1+e^{-x}} = \dfrac{1}{1+e^x} + \dfrac{e^x}{1+e^x} = 1.$

Consequently, $I + J = \displaystyle\int_0^1 \mathrm{d}x = 1.$

2° $J = \displaystyle\int_0^1 \frac{e^x\,\mathrm{d}x}{1+e^x} = [\ln(e^x+1)]_0^1 = \ln\frac{e+1}{2}$. Then $I = 1 - J = \ln\dfrac{2e}{e+1}$.

Exercise 4

1° Let $t = \tan\dfrac{x}{2}$. So $\mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$. It yields :

$$I = \int \frac{\mathrm{d}t}{4-t^2} = \frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| + c, \quad \text{where } t = \tan\frac{x}{2} \text{ and } c \in \mathbb{R}.$$

2° Integrating by parts, where $u = (\ln x)^2$ and $\mathrm{d}v = \mathrm{d}x$, we get :

$$J = \int (\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - 2\int \ln x\,\mathrm{d}x$$

If integrating by parts, once more, we obtain :

$$J = \int (\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - 2x\ln x + 2x + \text{cnst}.$$

Exercise 5

1° Reducing to the same denominator :

$$\frac{(bt+c)(1+t)+a(1+t^2)}{(1+t)(1+t^2)} = \frac{(a+b)t^2+(b+c)t+c+a}{(1+t)(1+t^2)}.$$

Hence the following system should be solved :

$$\begin{cases} a+b = 0 \\ b+c = 0 \\ c+a = 1 \end{cases} \iff \begin{cases} a = \dfrac{1}{2} \\ b = -\dfrac{1}{2} \\ c = \dfrac{1}{2} \end{cases}$$

Now simply we calculate the integral :

$$\int_0^1 \frac{\mathrm{d}t}{(1+t)(1+t^2)} = \frac{1}{2}\int_0^1 \left(\frac{1}{1+t} + \frac{1}{1+t^2} - \frac{t}{1+t^2}\right) \mathrm{d}t.$$

- $\displaystyle\int_0^1 \frac{1}{1+t}\,\mathrm{d}t = [\ln(1+t)]_0^1 = \ln 2.$
- $\displaystyle\int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t = [\arctan(t)]_0^1 = \frac{\pi}{4}.$
- $\displaystyle\int_0^1 \frac{t}{1+t^2}\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{2t\,\mathrm{d}t}{1+t^2} = \frac{1}{2}\left[\ln(1+t^2)\right]_0^1 = \frac{\ln 2}{2}.$

Finally, as the integral of a sum is the sum of integrals, we get :

$$\int_0^1 \frac{\mathrm{d}t}{(1+t)(1+t^2)} = \frac{1}{2}\left(\ln 2 + \frac{\pi}{4} - \frac{\ln 2}{2}\right) = \frac{\ln 2}{4} + \frac{\pi}{8}.$$

2° $t = \tan x \Rightarrow x = \arctan t \Rightarrow \mathrm{d}x = \dfrac{\mathrm{d}t}{1+t^2}$. Hence

$$\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}x}{1+\tan x} = \int_0^1 \frac{\mathrm{d}t}{(1+t)(1+t^2)} = \frac{\pi}{8} + \frac{\ln 2}{4}.$$

3° We have

$$\frac{\cos x}{\cos x + \sin x} = \frac{1}{2}\frac{\cos x + \sin x + \cos x - \sin x}{\cos x + \sin x} = \frac{1}{2}\left[1 + \frac{\cos x - \sin x}{\cos x + \sin x}\right].$$

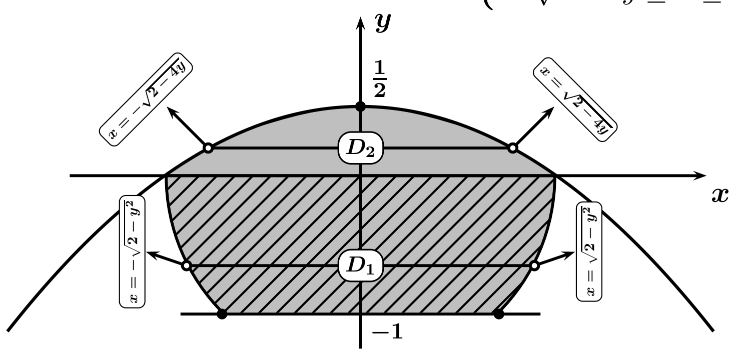
$$\therefore\ J = \frac{1}{2}\left[x + \ln|\cos x + \sin x|\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} + \frac{\ln 2}{4}.$$

Exercise 6

First method

$D = D_1 \cup D_2$, where

$$D_1 : \begin{cases} -1 \leq y \leq 0 \\ -\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2} \end{cases} \qquad D_2 : \begin{cases} 0 \leq y \leq \dfrac{1}{2} \\ -\sqrt{2-4y} \leq x \leq \sqrt{2-4y}. \end{cases}$$



$$\therefore I = \int_{-1}^{0} \mathrm{d}y \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)\,\mathrm{d}x + \int_{0}^{\frac{1}{2}} \mathrm{d}y \int_{-\sqrt{2-4y}}^{\sqrt{2-4y}} f(x,y)\,\mathrm{d}x.$$

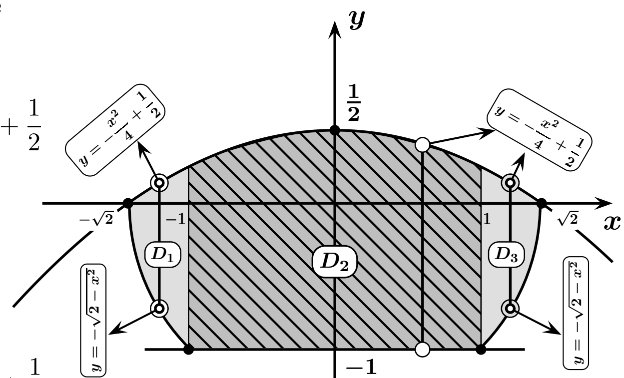
Second method

$D = D_1 \cup D_2 \cup D_3$, where

$$D_1 : \begin{cases} -\sqrt{2} \leq x \leq -1 \\ -\sqrt{2-x^2} \leq y \leq -\dfrac{x^2}{4} + \dfrac{1}{2} \end{cases}$$

$$D_2 : \begin{cases} -1 \leq x \leq 1 \\ -1 \leq x \leq -\dfrac{x^2}{4} + \dfrac{1}{2} \end{cases}$$

$$D_3 : \begin{cases} 1 \leq x \leq \sqrt{2} \\ -\sqrt{2-x^2} \leq x \leq -\dfrac{x^2}{4} + \dfrac{1}{2} \end{cases}$$

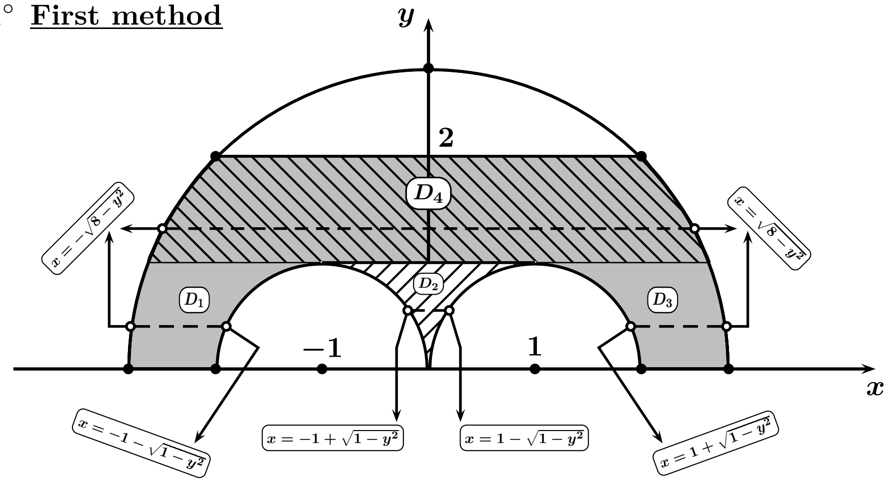


Therefore

$$I = \int_{-\sqrt{2}}^{-1} \mathrm{d}x \int_{-\sqrt{2-x^2}}^{-\frac{x^2}{4}+\frac{1}{2}} f(x,y)\,\mathrm{d}y + \int_{-1}^{1} \mathrm{d}x \int_{-1}^{-\frac{x^2}{4}+\frac{1}{2}} f(x,y)\,\mathrm{d}y + \int_{1}^{\sqrt{2}} \mathrm{d}x \int_{-\sqrt{2-x^2}}^{-\frac{x^2}{4}+\frac{1}{2}} f(x,y)\,\mathrm{d}y.$$

Exercise 7

1° **First method**



We may write : $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where

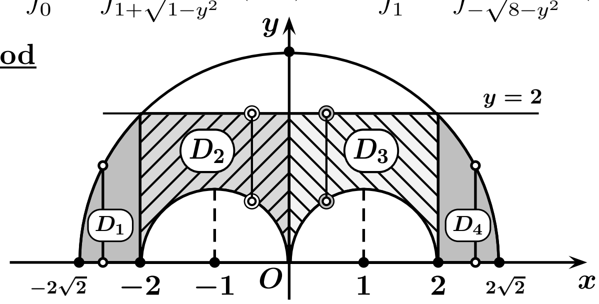
$$D_1 : \begin{cases} 0 \le y \le 1 \\ -\sqrt{8-y^2} \le x \le -1-\sqrt{1-y^2} \end{cases} \qquad D_2 : \begin{cases} 0 \le y \le 1 \\ -1+\sqrt{1-y^2} \le x \le 1-\sqrt{1-y^2} \end{cases}$$

$$D_3 : \begin{cases} 0 \le y \le 1 \\ 1+\sqrt{1-y^2} \le x \le \sqrt{8-y^2} \end{cases} \qquad D_4 : \begin{cases} 1 \le y \le 2 \\ -\sqrt{8-y^2} \le x \le \sqrt{8-y^2} \end{cases}$$

Then, $I = \int_0^1 \mathrm{d}y \int_{-\sqrt{8-y^2}}^{-1-\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x + \int_0^1 \mathrm{d}y \int_{-1+\sqrt{1-y^2}}^{1-\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x$

$$+ \int_0^1 \mathrm{d}y \int_{1+\sqrt{1-y^2}}^{\sqrt{8-y^2}} f(x,y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{-\sqrt{8-y^2}}^{\sqrt{8-y^2}} f(x,y)\,\mathrm{d}x.$$

Second method



We may write : $D = D_1 \cup D_2 \cup D_3 \cup D_4$ with

$$D_1 : \begin{cases} -2\sqrt{2} \le x \le -2 \\ 0 \le y \le \sqrt{8-x^2} \end{cases} \qquad D_2 : \begin{cases} -2 \le y \le 0 \\ \sqrt{-2x-x^2} \le y \le 2 \end{cases}$$

$$D_3 : \begin{cases} 0 \le x \le 2 \\ \sqrt{2x-x^2} \le y \le 2 \end{cases} \qquad D_4 : \begin{cases} 2 \le x \le 2\sqrt{2} \\ 0 \le y \le \sqrt{8-x^2} \end{cases}$$

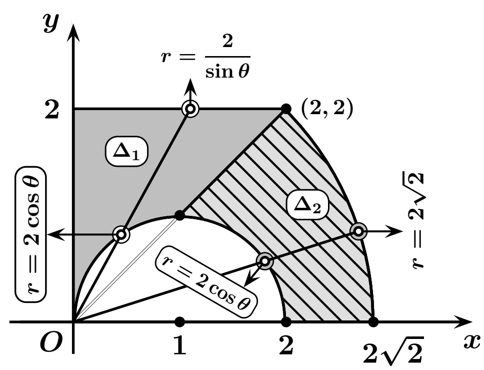
Therefore

$$\begin{aligned} I \;=\; & \int_{-2\sqrt{2}}^{-2} \mathrm{d}x \int_{0}^{\sqrt{8-x^2}} f(x,y)\,\mathrm{d}y + \int_{-2}^{0} \mathrm{d}x \int_{\sqrt{-2x-x^2}}^{2} f(x,y)\,\mathrm{d}y \\ & + \int_{0}^{2} \mathrm{d}x \int_{\sqrt{2x-x^2}}^{2} f(x,y)\,\mathrm{d}y + \int_{2}^{2\sqrt{2}} \mathrm{d}x \int_{0}^{\sqrt{8-x^2}} f(x,y)\,\mathrm{d}y. \end{aligned}$$

2° $\Delta = \Delta_1 \cup \Delta_2$ where

$$\Delta_1 = \begin{cases} 0 \le \theta \le \dfrac{\pi}{4} \\ 2\cos\theta \le r \le 2\sqrt{2} \end{cases}$$

$$\Delta_2 = \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2} \\ 2\cos\theta \le r \le \dfrac{2}{\sin\theta} \end{cases}$$



$\mathbf{Area}(\Delta) = \mathbf{Area}(\Delta_1) + \mathbf{Area}(\Delta_2)$, where

$$\begin{aligned} \mathbf{Area}(\Delta_1) \;=\; & \int_{0}^{\frac{\pi}{4}} \mathrm{d}\theta \int_{2\cos\theta}^{2\sqrt{2}} r\,\mathrm{d}r = \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \Big(8 - 4\cos^2\theta\Big)\,\mathrm{d}\theta \\ =\; & \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \left(8 - 4\frac{1+\cos(2\theta)}{2}\right)\mathrm{d}\theta = \frac{1}{2}\Big[6\theta - \sin(2\theta)\Big]_{0}^{\frac{\pi}{4}} \\ =\; & \frac{3\pi}{4} - \frac{1}{2}. \end{aligned}$$

And

$$\begin{aligned} \mathbf{Area}(\Delta_2) \;=\; & \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{2\cos\theta}^{\frac{2}{\sin\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{4}{\sin^2\theta} - 4\cos^2\theta\right)\mathrm{d}\theta \\ =\; & \frac{1}{2}\Big[-4\cot(\theta) - 2\theta - \sin(2\theta)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ =\; & \frac{5}{2} - \frac{\pi}{4}. \end{aligned}$$

Finally : $\mathbf{Area}(\Delta) = \dfrac{\pi}{2} + 2.$

Exercise 8

1° (a) By direct verification :

$$\frac{1}{x(x^2-1)} = -\frac{(x^2-1)-x^2}{x(x^2-1)} = -\frac{1}{x} + \frac{x}{x^2-1}.$$

(b) We are dealing with a first order linear differential equation. Looking for its solution in the form $y = uv$, it yields :

$$u'v + \left[v' + \frac{2}{x(x^2-1)}v\right] u = \frac{x^2}{x^2-1}.$$

Choose v such that :

$$v' + \frac{2}{x(x^2-1)}v = 0 \iff \frac{\mathrm{d}v}{v} = -\frac{2\,\mathrm{d}x}{x(x^2-1)}.$$

Hence by integration, we obtain :

$$\ln v = \ln\frac{x^2}{x^2-1} \quad \Rightarrow \quad v = \frac{x^2}{x^2-1}.$$

Finally, $u'v = \dfrac{x^2}{x^2-1}$ gives $u' = 1$, so $u = x + c$. As a result, the general solution of the considered equation is :

$$y = \frac{x^2}{x^2-1}\,(x+c)\,, \qquad c \in \mathbb{R}.$$

2° The differential equation $xy' + \sqrt{xy}\,y' = y$ is written as :

$$y' = \frac{y}{x+\sqrt{xy}} = \frac{\dfrac{y}{x}}{1+\sqrt{\dfrac{y}{x}}},$$

which is a homogeneous differential equation. Thus, let $z = \dfrac{y}{x}$, we reach $y' = xz' + z$, and consequently :

$$xz' + z = \frac{z}{1+\sqrt{z}} \iff \frac{1+\sqrt{z}}{z\sqrt{z}}\,\mathrm{d}z = -\frac{\mathrm{d}x}{x}.$$

By integration we get :

$$\int \frac{1+\sqrt{z}}{z\sqrt{z}}\,\mathrm{d}z = -\int\frac{\mathrm{d}x}{x} \;\Leftrightarrow\; \int\left(\frac{1}{z\sqrt{z}} + \frac{1}{z}\right)\mathrm{d}z = -\int\frac{\mathrm{d}x}{x}.$$

Hence $-\dfrac{2}{\sqrt{z}} + \ln z = -\ln x + c$, where $c \in \mathbb{R}$ and $z = \dfrac{y}{x}$.

LEBANESE UNIVERSITY Date: 19 February 2013
Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 114

Exercise 1 **[10 points]**

Consider

$$f(x) = x^2\left(\sqrt{3+\cos\frac{1}{x}} - \ln\left(2 - e^{\frac{1}{x}}\right) + a + b\sin\frac{2}{x}\right)$$

where a and b are real numbers. Determine a and b such that $\lim\limits_{x\to+\infty} f(x)$ is finite. Determine then then this limit.

Exercise 2 **[10 points]**

Let

$$f(x) = \sqrt{x^2-1}\,\arcsin\frac{1}{x}.$$

Set $x = \dfrac{1}{t}$.

1° Give the f.e. to order 3 of $tf\left(\dfrac{1}{t}\right)$ in neighborhood of $t = 0_-$.

2° Deduce the equation of the asymptote (D) to the curve (C) of f at $-\infty$, then precise the relative position of (D) and (C) at $-\infty$.

Exercise 3 **[14 points]**

1° (a) Calculate $I(t) = \displaystyle\int \frac{2}{(1+t)(1+t^2)}\,\mathrm{d}t$.

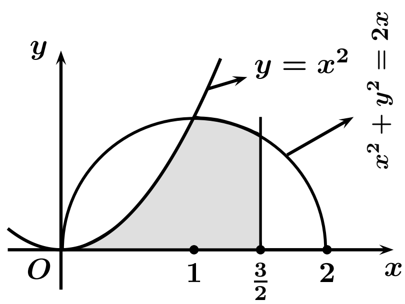
(b) Calculate $J(x) = \displaystyle\int \frac{x}{(1+x^2)^2}\ln(1+x)\,\mathrm{d}x$.

2° (a) Calculate $L(t) = \displaystyle\int \frac{2t^2}{(1+t^2)(3+t^2)}\,\mathrm{d}t$.

(b) Calculate $M(x) = \displaystyle\int \frac{\sin x\tan\frac{x}{2}}{2+\cos x}\,\mathrm{d}x$.

Exercise 4 **[10 points]**

Consider the shaded domain D in the following figure :



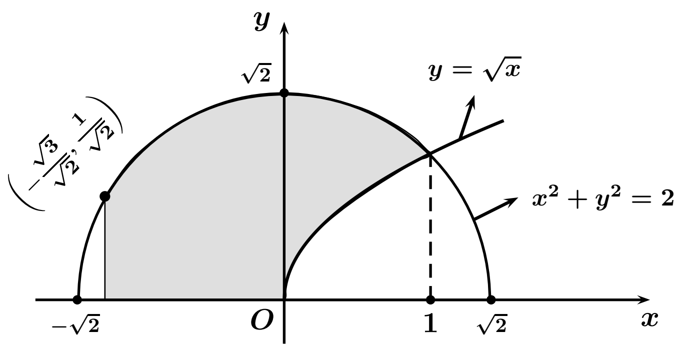
Write, in two distinct forms, the double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

as a succession of two simple integrals, by using **cartesian coordinates**.

Exercise 5 **[16 points]**

Consider the shaded domain D in the following figure :



By using **polar coordinates**, write down the double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

as a succession of two simple integrals.

Exercise 6 **[10 points]**

Solve the following differential equations :

1° $xy' = y + \sqrt{x^2 + xy}$ (where $x > 0$).

2° $y' + xy = x$.

Exercise 1

Let $x = \dfrac{1}{t}$. Then we have $x \to +\infty \Leftrightarrow t \to 0_+$. It results :

$$f\left(\frac{1}{t}\right) = \frac{\sqrt{3+\cos t} - \ln(2-e^t) + a + b\sin(2t)}{t^2}.$$

Expanding in neighborhood of $t = 0$ to order 2 :

$$\begin{aligned}
\sqrt{3+\cos t} &= \sqrt{4 - \frac{t^2}{2}} + t^2\varepsilon(t) = 2\sqrt{1 - \frac{t^2}{8}} + t^2\varepsilon(t) = 2 - \frac{t^2}{8} + t^2\varepsilon(t);\\
\ln(2-e^t) &= \ln\left(1 + \left(-t - \frac{t^2}{2}\right)\right) + t^2\varepsilon(t) = -t - t^2 + t^2\varepsilon(t);\\
\sin(2t) &= 2t + t^2\varepsilon(t).
\end{aligned}$$

Thus,

$$f\left(\frac{1}{t}\right) = \frac{2+a}{t^2} + \frac{2b+1}{t} + \frac{7}{8} + \varepsilon(t).$$

For $a = -2$ and $b = -\dfrac{1}{2}$, we get $\lim\limits_{x\to+\infty} f(x) = \dfrac{7}{8}$.

Exercise 2

1° $tf\left(\dfrac{1}{t}\right) = -\sqrt{1-t^2}\arcsin(t)$. To order 3 near $t = 0_-$:

- $\sqrt{1-t^2} = 1 - \dfrac{t^2}{2} + t^2\varepsilon(t)$.
- $\arcsin(t) = t + \dfrac{t^3}{6} + t^3\varepsilon(t)$.

Thus :

$$\begin{aligned}
tf\left(\frac{1}{t}\right) &= -\left(1 - \frac{t^2}{2}\right)\left(t + \frac{t^3}{6}\right) + t^3\varepsilon(t)\\
&= -t + \frac{t^3}{3} + t^3\varepsilon(t).
\end{aligned}$$

2° We have

$$f(x) = -1 + \frac{1}{3x^2} + \frac{1}{x^2}\varepsilon(x).$$

It results that the asymptote (D) to the curve (C) of f at $-\infty$ is of equation $y = -1$, and since $f(x) - y \underset{-\infty}{\simeq} \dfrac{1}{3x^2} > 0$ so (C) is situated above (D) at $-\infty$.

Exercise 3

1° (a) Simply we have :

$$\frac{2}{(1+t)(1+t^2)} = \frac{1}{1+t} + \frac{1-t}{1+t^2}.$$

Thus : $I(t) = \ln|1+t| + \arctan t - \dfrac{1}{2}\ln(1+t^2) + \text{cnst}.$

(b) Integrating by parts, where

$$\begin{cases} u = \ln(1+x) \\ \mathrm{d}v = \dfrac{x}{(1+x^2)^2}\,\mathrm{d}x \end{cases} \quad \text{so} \quad \begin{cases} \mathrm{d}u = \dfrac{\mathrm{d}x}{1+x}; \\ v = -\dfrac{1}{2(1+x^2)}. \end{cases}$$

Hence, $\begin{aligned} J(x) &= uv - \int v\,\mathrm{d}u = -\frac{\ln(1+x)}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{(1+x)(1+x^2)}\,\mathrm{d}x \\ &= -\frac{\ln(1+x)}{2(1+x^2)} + \frac{1}{4}I(x). \end{aligned}$

2° (a) $\dfrac{2t^2}{(1+t^2)(3+t^2)} = \dfrac{3}{3+t^2} - \dfrac{1}{1+t^2}.$

Hence, $L(t) = \dfrac{3}{\sqrt{3}}\arctan\dfrac{t}{\sqrt{3}} - \arctan t + \text{cnst}.$

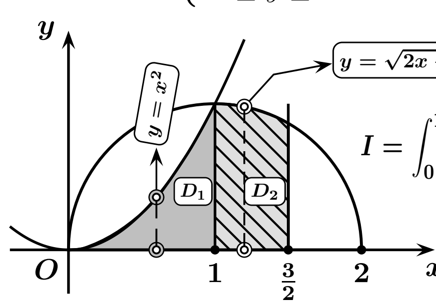
(b) Let $t = \tan\dfrac{x}{2}$. We get $x = 2\arctan t$ and $\mathrm{d}x = 2\dfrac{\mathrm{d}t}{1+t^2}$. Hence

$$\begin{aligned} M(x) &= \int \frac{\frac{2t}{1+t^2}t}{2+\frac{1-t^2}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} \\ &= 4\int \frac{t^2}{(1+t^2)(3+t^2)}\,\mathrm{d}t = 2L(t) = 2\sqrt{3}\arctan\frac{\arctan\frac{x}{2}}{\sqrt{3}} - x + \text{cnst}. \end{aligned}$$

Exercise 4

First method : Vertically (i.e. fixing x) : $D = D_1 \cup D_2$ where

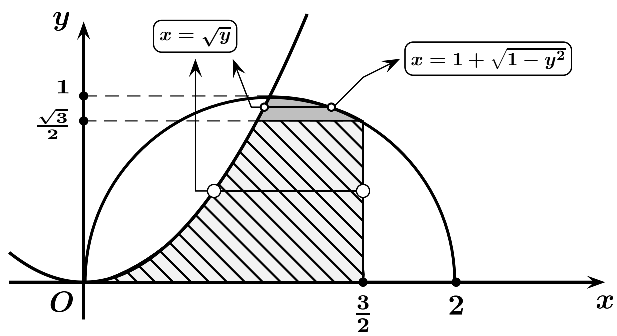
$$D_1 : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{cases} \quad \text{and} \quad D_2 : \begin{cases} 1 \le x \le \dfrac{3}{2} \\ 0 \le y \le \sqrt{2x - x^2} \end{cases}$$



$$I = \int_0^1 \mathrm{d}x \int_0^{x^2} f(x,y)\,\mathrm{d}y + \int_1^{\frac{3}{2}} \mathrm{d}x \int_0^{\sqrt{2x-x^2}} f(x,y)\,\mathrm{d}y.$$

<u>Second method :</u> Horizontally (i.e.fixing y) : $D = D_1 \cup D_2$ where

$$D_1 : \begin{cases} 0 \le y \le \dfrac{\sqrt{3}}{2} \\ \sqrt{y} \le x \le \dfrac{3}{2} \end{cases} \quad \text{and} \quad D_2 : \begin{cases} \dfrac{\sqrt{3}}{2} \le y \le 1 \\ \sqrt{y} \le x \le 1 + \sqrt{1-y^2} \end{cases}$$

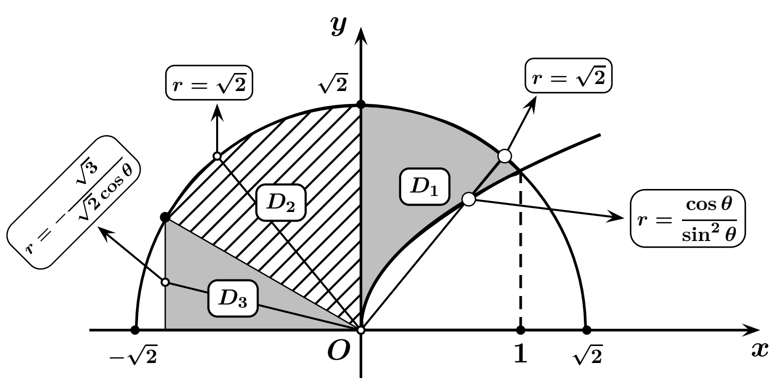


Hence, $I = \displaystyle\int_0^{\frac{\sqrt{3}}{2}} \mathrm{d}y \int_{\sqrt{y}}^{\frac{3}{2}} f(x,y)\,\mathrm{d}x + \int_{\frac{\sqrt{3}}{2}}^{1} \mathrm{d}y \int_{\sqrt{y}}^{1+\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x.$

Exercise 5

By polar coordinates : $D = D_1 \cup D_2 \cup D_3$, where

$$D_1 : \begin{cases} \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2} \\ \dfrac{\cos\theta}{\sin^2\theta} \le r \le \sqrt{2} \end{cases} \quad D_2 : \begin{cases} \dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6} \\ 0 \le r \le \sqrt{2} \end{cases} \quad D_3 : \begin{cases} \dfrac{5\pi}{6} \le \theta \le \pi \\ 0 \le r \le \dfrac{-\sqrt{3}}{\sqrt{2}\cos\theta} \end{cases}$$



Therefore

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{\cos\theta}{\sin^2\theta}}^{\sqrt{2}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \mathrm{d}\theta \int_0^{\sqrt{2}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r + \int_{\frac{5\pi}{6}}^{\pi} \mathrm{d}\theta \int_0^{\frac{-\sqrt{3}}{\sqrt{2}\cos\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r.$$

Exercise 6

1° The given differential equation is written as :

$$y' = \frac{y}{x} + \sqrt{1 + \frac{y}{x}} \qquad (*)$$

Which is a homogeneous equation. Let $z = \dfrac{y}{x}$ thus, $y' = xz' + z$ and

$$xz' = \sqrt{1+z} \quad \Rightarrow \quad \frac{\mathrm{d}z}{\sqrt{1+z}} = \frac{\mathrm{d}x}{x}.$$

Integrating, we obtain :

$$2\sqrt{1+z} = \ln x + \text{cnst} \quad \Rightarrow \quad 2\sqrt{1 + \frac{y}{x}} = \ln x + \text{cnst}.$$

2° $y' + xy = x$ is a linear differential equation of first order. Set $y = uv$. Then

$$u'v + [v' + xv]\, u = x.$$

- **Finding v :** choose v solution of $v' + xv = 0$, we get $\dfrac{\mathrm{d}v}{v} = -x$ and thus $\ln v = -\dfrac{x^2}{2}$. So $v = e^{-\frac{x^2}{2}}$.
- **Finding u :** The function u is a solution of $u'v = x$.

  Thus $u = \displaystyle\int e^{\frac{x^2}{2}} x \,\mathrm{d}x = e^{\frac{x^2}{2}} + c$, where $c \in \mathbb{R}$.

Finally, the general solution of the given differential equation is

$$y = \left(e^{\frac{x^2}{2}} + c\right) e^{-\frac{x^2}{2}} = 1 + ce^{-\frac{x^2}{2}}.$$

LEBANESE UNIVERSITY Date: 03 February 2014
Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 120

Exercise 1 **(10 points)**

1° Solve the following differential equation : $\left(2+\frac{y}{x}e^{\frac{y}{x}}\right)y' = \frac{y^2}{x^2}e^{\frac{y}{x}}$.

2° Solve the differential equation $(x+1)y' + xy = e^{-x}$ and determine the particular solution that verifies $y(0) = -1$.

Exercise 2 **(12 points)**

1° Calculate the integral $I(x) = \int e^{-x}\ln(2+e^x)\,\mathrm{d}x$.

2° Calculate $J(t) = \int \frac{t}{t^2-2t+5}\,\mathrm{d}t$ then $K(x) = \int \frac{1}{3+x-\sqrt{1+2x}}\,\mathrm{d}x$.

3° Calculate $M(x) = \int \frac{\sin(2x)}{(1+\tan x)\cos^4 x}\,\mathrm{d}x$.

Exercise 3 **(8 points)**

Let f be a function defined by

$$f(x) = x^3\left[e^{\frac{2x}{1+x^2}} + \cos\left(\frac{a}{x}\right) + \frac{b}{x} - 2\right].$$

Determine the real constants a and b, so that $\lim_{x\to+\infty} f(x)$ is finite. Find then this limit.

Exercise 4 **(12 points)**

Given the function f defined by $f(x) = \frac{(\cos x)^{\frac{2}{x}} - 7x^2/2}{\sqrt[3]{1+3\sin x}}$.

1° Give the f.e. to order 3 of $f(x)$ near 0.

2° Show that f could be extended by continuity to $x=0$. Set g as its extension.

3° Determine $g'(0)$.

4° Determine the equation of the tangent (T) to the curve (C) of g at the point of abscissa $x=0$, and precise the relative positions of the two curves in a neighborhood of $x=0$.

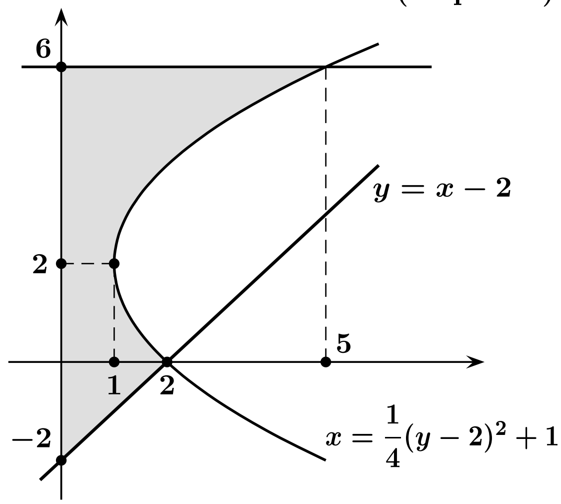
Exercise 5 **(12 points)**

Consider the shaded domain D in the adjacent figure.

Write down, in two different forms, the following double integral

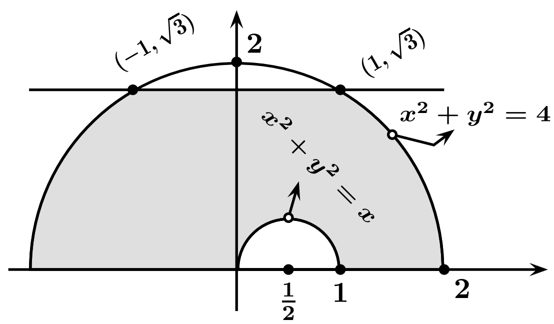
$$I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

as a succession of two simple integrals, by using **the cartesian coordinates**.



Exercise 6 **(16 points)**

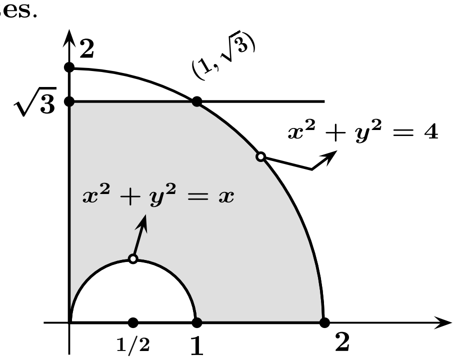
1° Consider the shaded domain D in the figure below :



Write down, in two different forms, the expression of the double integral $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ as a succession of two simple integrals, by using **the cartesian coordinates**.

2° Consider the shaded domain D in the adjacent figure.

Write down the double double $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ as a succession of two simple integrals, by using **polar coordinates**.



Exercise 1

1° The considered differential equation is written as :

$$y' = \frac{\frac{y^2}{x^2}e^{\frac{y}{x}}}{2+\frac{y}{x}e^{\frac{y}{x}}} \tag{1}$$

It's a homogeneous diff. equation. Let $z = \dfrac{y}{x}$. Then, $y' = xz' + z$ and the diff. equation (1) is rewritten as :

$$xz' + z = \frac{z^2e^z}{2+ze^z} \Leftrightarrow xz' = -\frac{2z}{2+ze^z} \Leftrightarrow -\frac{2+ze^z}{2z}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By integration, we get

$$-\int\left(\frac{1}{z}+\frac{1}{2}e^z\right)\mathrm{d}z = \int\frac{\mathrm{d}x}{x}, \quad \text{hence} \quad -\ln|z| - \frac{1}{2}e^z = \ln|x| + c.$$

Thus : $\ln\left|\dfrac{y}{x}\right| + \dfrac{1}{2}e^{\frac{y}{x}} = -\ln|x| + c$ where $c \in \mathbb{R}$.

2° The considered differential equation is written as :

$$y' + \frac{x}{x+1}y = \frac{1}{x+1}e^{-x} \tag{$*$}$$

Let $y = uv$, then $y' = u'v + v'u$ and the diff. equation $(*)$ becomes :

$$u'v + u\left[v' + \frac{x}{x+1}v\right] = \frac{1}{x+1}e^{-x} \tag{$**$}$$

We choose v such that

$$v' + \frac{x}{x+1}v = 0 \Leftrightarrow \frac{\mathrm{d}v}{v} = -\left[1 - \frac{1}{x+1}\right]\mathrm{d}x.$$

We get by direct integration :

$$\ln v = -x + \ln(x+1) \qquad \text{then} \qquad v = (x+1)e^{-x}.$$

From $(**)$, we have $u'v = \dfrac{1}{x+1}e^{-x}$ then $u' = \dfrac{1}{(x+1)^2}$. By integrating, it yields : $u = -\dfrac{1}{x+1} + c$ where $c \in \mathbb{R}$. Thus

$$y = \left(-\frac{1}{x+1} + c\right)(x+1)e^{-x}.$$

Finally $y(0) = -1$ gives $-1 = (-1+c)$ so $c = 0$. Therefore, $y = -e^{-x}$.

Exercise 2

1° Let $t = e^x$. It yields $x = \ln t$ and $\mathrm{d}x = \dfrac{\mathrm{d}t}{t}$. Thus

$$I = \int \ln(2+t)\frac{\mathrm{d}t}{t^2}.$$

Integrating by parts with $u = \ln(2+t)$ and $\mathrm{d}v = \dfrac{\mathrm{d}t}{t^2}$. Then

$$\mathrm{d}u = \frac{\mathrm{d}t}{2+t}, \quad \text{and} \quad v = -\frac{1}{t}.$$

Hence :

$$I = uv - \int v\,\mathrm{d}u = -\frac{1}{t}\ln(2+t) + \int \frac{1}{t(2+t)}\,\mathrm{d}t.$$

But $\dfrac{1}{(2+t)t} = \dfrac{1}{2t} - \dfrac{1}{2(2+t)}$. Thus,

$$\int \frac{1}{t(2+t)}\,\mathrm{d}t = \frac{1}{2}\ln(t) - \frac{1}{2}\ln(2+t) + \text{cnst}.$$

Consequently, $I = -e^{-x}\ln(2+e^x) + \dfrac{x}{2} - \dfrac{1}{2}\ln(2+e^x) + \text{cnst}.$

2° •
$$\begin{aligned} J(t) &= \int \frac{t}{t^2-2t+5}\,\mathrm{d}t = \frac{1}{2}\int \frac{2t-2+2}{t^2-2t+5}\,\mathrm{d}t \\ &= \frac{1}{2}\int \frac{2t-2}{t^2-2t+5}\,\mathrm{d}t + \int \frac{1}{t^2-2t+5}\,\mathrm{d}t = J_1 + J_2. \end{aligned}$$

▷ $J_1 = \dfrac{1}{2}\displaystyle\int \frac{2t-2}{t^2-2t+5}\,\mathrm{d}t = \frac{1}{2}\ln\left|t^2-2t+5\right| + \text{cnst}.$

▷ $J_2 = \displaystyle\int \frac{\mathrm{d}t}{t^2-2t+5} = \int \frac{\mathrm{d}(t-1)}{2^2+(t-1)^2} = \frac{1}{2}\arctan\frac{t-1}{2} + \text{cnst}.$

Thus, $J(t) = \dfrac{1}{2}\ln\left|t^2-2t+5\right| + \dfrac{1}{2}\arctan\dfrac{t-1}{2} + \text{cnst}.$

• Let $t = \sqrt{1+2x}$ thus, $x = \dfrac{1}{2}(t^2-1)$ and $\mathrm{d}x = t\,\mathrm{d}t$. Hence

$$K(x) = \int \frac{t}{3+\frac{t^2-1}{2}-t}\,\mathrm{d}t = 2\int \frac{t}{5-2t+t^2}\,\mathrm{d}t = 2J(t)$$

where $t = \sqrt{1+2x}$.

3° $M(x) = 2\displaystyle\int \frac{\tan x}{(1+\tan x)\cos^2 x}\,\mathrm{d}x$. Set $t = \tan x$:

$$\begin{aligned} M(x) &= 2\int \frac{t}{1+t}\,\mathrm{d}t = 2\int \left(1 - \frac{1}{1+t}\right)\mathrm{d}t \\ &= 2t - 2\ln|1+t| + \text{cnst} \quad \text{where} \quad t = \tan x. \end{aligned}$$

Exercise 3

Let $x = 1/t$ and give the f.e. of $f(1/t)$ to order 0 in neighborhood of $t = 0$. We have :

$$f\left(\frac{1}{t}\right) = \frac{1}{t^3}\left[e^{\frac{2t}{1+t^2}} + \cos(at) + bt - 2\right].$$

In neighborhood of $t = 0$ and to order 3 :

- $$\begin{aligned} e^{\frac{2t}{1+t^2}} &= e^{2t-2t^3} + t^3\varepsilon(t) \\ &= 1 + 2t - 2t^3 + \frac{1}{2}(2t-2t^3)^2 + \frac{1}{6}(2t-2t^3)^3 + t^3\varepsilon(t) \\ &= 1 + 2t + 2t^2 - \frac{2}{3}t^3 + t^3\varepsilon(t). \end{aligned}$$

- $\cos(at) = 1 - \dfrac{a^2t^2}{2} + t^3\varepsilon(t).$

It follows that

$$f\left(\frac{1}{t}\right) = \frac{b+2}{t^2} + \frac{4-a^2}{2t} - \frac{2}{3} + \varepsilon(t).$$

The limit is finite iff : $a = \pm 2$ and $b = -2$. In this case : $\lim\limits_{x \to +\infty} f(x) = -\dfrac{2}{3}$.

Exercise 4

1° $(\cos x)^{\frac{2}{x}} = e^{\frac{2}{x}\ln(\cos x)}$. In neighborhood of $x = 0$ and to order 4

$$\begin{aligned} \ln(\cos x) &= \ln\left[1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right] + x^4\varepsilon(x) \\ &= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + x^4\varepsilon(x) \\ &= -\frac{x^2}{2} - \frac{x^4}{12} + x^4\varepsilon(x). \end{aligned}$$

Hence

$$\begin{aligned} (\cos x)^{\frac{2}{x}} &= e^{-x-\frac{x^3}{6}} + x^3\varepsilon(x) \\ &= 1 - x - \frac{x^3}{6} + \frac{1}{2}\left(-x - \frac{x^3}{6}\right)^2 + \frac{1}{6}\left(-x - \frac{x^3}{6}\right)^3 + x^3\varepsilon(x) \\ &= 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + x^3\varepsilon(x). \end{aligned}$$

On the other hand, being always near $x=0$:

$$\begin{aligned}\sqrt[3]{1+3\sin x} &= \left(1+3x-\frac{x^3}{2}\right)^{\frac{1}{3}}+x^3\varepsilon(x)\\ &= 1+\frac{1}{3}\left(3x-\frac{x^3}{2}\right)-\frac{1}{9}\left(3x-\frac{x^3}{2}\right)^2+\frac{5}{81}\left(3x-\frac{x^3}{2}\right)^3+x^3\varepsilon(x)\\ &= 1+x-x^2+\frac{3}{2}x^3+x^3\varepsilon(x).\end{aligned}$$

Thus

$$f(x)=\frac{1-x-3x^2-\frac{1}{3}x^3}{1+x-x^2+\frac{3}{2}x^3}+x^3\varepsilon(x)=1-2x-\frac{23}{6}x^3+x^3\varepsilon(x).$$

2° $\lim_{x\to 0} f(x)=1$ hence f is extendable by continuity to $x=0$.

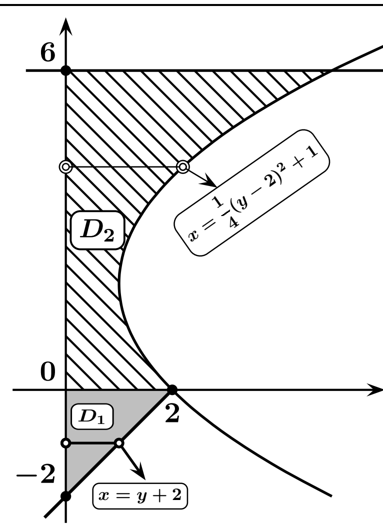
3° $g'(0)=-2$

4° $y_T=1-2x$.

- $f(x)-y_T \underset{0_+}{\simeq} -\frac{23}{6}x^3<0$ and (C) is below the tangent (T).
- $f(x)-y_T \underset{0_-}{\simeq} -\frac{23}{6}x^3>0$ and (C) is above the tangent (T).

Exercise 5

Horizontally (by fixing y) :



$D=D_1\cup D_2$ with

$$D_1:\begin{cases}-2\le y\le 0\\ 0\le x\le y+2\end{cases}$$

$$D_2:\begin{cases}0\le y\le 6\\ 0\le x\le \frac{1}{4}(y-2)^2+1\end{cases}$$

Thus,

$$\boldsymbol{I=\int_{-2}^{0}\mathrm{d}y\int_{0}^{y+2}f(x,y)\,\mathrm{d}x+\int_{0}^{6}\mathrm{d}y\int_{0}^{\frac{1}{4}(y-2)^2+1}f(x,y)\,\mathrm{d}x.}$$

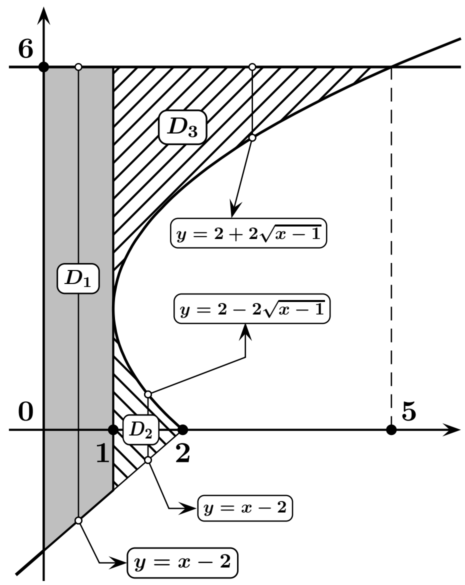
Vertically (by fixing x) :

$D = D_1 \cup D_2 \cup D_3$ with

$D_1 : \begin{cases} 0 \leq x \leq 1 \\ x-2 \leq y \leq 6 \end{cases}$

$D_2 : \begin{cases} 1 \leq x \leq 2 \\ x-2 \leq y \leq 2-2\sqrt{x-1} \end{cases}$

$D_3 : \begin{cases} 1 \leq x \leq 5 \\ 2+2\sqrt{x-1} \leq y \leq 6 \end{cases}$



Thus

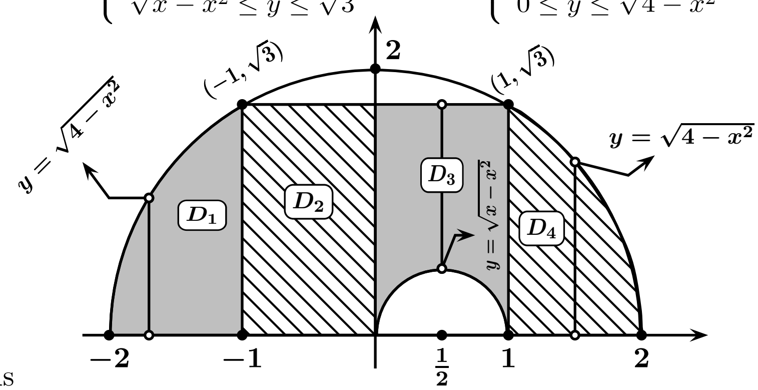
$$I = \int_0^1 \mathrm{d}x \int_{x-2}^{6} f(x,y)\,\mathrm{d}y + \int_1^2 \mathrm{d}x \int_{x-2}^{2-2\sqrt{x-1}} f(x,y)\,\mathrm{d}y + \int_1^5 \mathrm{d}x \int_{2+2\sqrt{x-1}}^{6} f(x,y)\,\mathrm{d}y.$$

Exercise 6

1° **Vertically (by fixing x) :** $D = D_1 \cup D_2 \cup D_3 \cup D_4$ with

$D_1 : \begin{cases} -2 \leq x \leq -1 \\ 0 \leq y \leq \sqrt{4-x^2} \end{cases}$ $\quad D_2 : \begin{cases} -1 \leq x \leq 0 \\ 0 \leq y \leq \sqrt{3} \end{cases}$

$D_3 : \begin{cases} 0 \leq x \leq 1 \\ \sqrt{x-x^2} \leq y \leq \sqrt{3} \end{cases}$ $\quad D_4 : \begin{cases} 1 \leq x \leq 2 \\ 0 \leq y \leq \sqrt{4-x^2} \end{cases}$



Thus

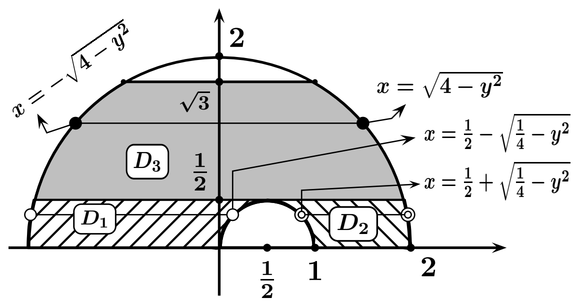
$$I = \int_{-2}^{-1} \mathrm{d}x \int_0^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y + \int_{-1}^{0} \mathrm{d}x \int_0^{\sqrt{3}} f(x,y)\,\mathrm{d}y + \int_0^1 \mathrm{d}x \int_{\sqrt{x-x^2}}^{\sqrt{3}} f(x,y)\,\mathrm{d}y + \int_1^2 \mathrm{d}x \int_0^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y.$$

Horizontally (by fixing y) : $D = D_1 \cup D_2 \cup D_3$ with

$$D_1 : \begin{cases} 0 \le y \le \dfrac{1}{2} \\ -\sqrt{4-y^2} \le x \le \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - y^2} \end{cases}$$

$$D_2 : \begin{cases} 0 \le y \le \dfrac{1}{2} \\ \dfrac{1}{2} + \sqrt{\dfrac{1}{4} - y^2} \le x \le \sqrt{4-y^2} \end{cases}$$

$$D_3 : \begin{cases} \dfrac{1}{2} \le y \le \sqrt{3} \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \end{cases}$$

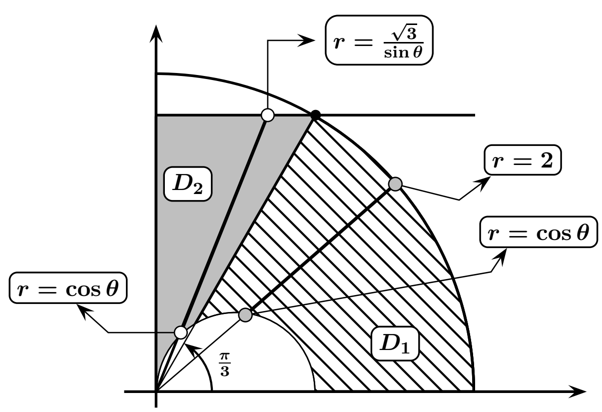


Thus

$$I = \int_0^{\frac{1}{2}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{\frac{1}{2}-\sqrt{\frac{1}{4}-y^2}} f(x,y)\,\mathrm{d}x + \int_0^{\frac{1}{2}} \mathrm{d}y \int_{\frac{1}{2}+\sqrt{\frac{1}{4}-y^2}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{\frac{1}{2}}^{\sqrt{3}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x.$$

2° By polar coordinates : $D = D_1 \cup D_2$ with

$$D_1 : \begin{cases} 0 \le \theta \le \dfrac{\pi}{3} \\ \cos\theta \le r \le 2 \end{cases} \qquad D_2 : \begin{cases} \dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{2} \\ \cos\theta \le r \le \dfrac{\sqrt{3}}{\sin\theta} \end{cases}$$



Thus

$$I = \int_0^{\frac{\pi}{3}} \mathrm{d}\theta \int_{\cos\theta}^{2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\cos\theta}^{\frac{\sqrt{3}}{\sin\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r.$$

LEBANESE UNIVERSITY Date: 13 February 2015

Faculty of Sciences (I)

Final Exam **Math171** Duration: 2 hours

ANSWER PAGE 128

Exercise 1 **(14 points)**

1° (a) Calculate $I(x) = \displaystyle\int \frac{x}{2 - x^2 + 2x}\,\mathrm{d}x$.

(b) Calculate $J(t) = \displaystyle\int \frac{\mathrm{d}t}{1 + t + 2\sqrt{1-t}}$.

2° (a) Calculate $I(t) = \displaystyle\int \frac{1}{(1+t)(2+t)}\,\mathrm{d}t$.

(b) Calculate $J(t) = \displaystyle\int \frac{\ln(1+t)}{(2+t)^2}\,\mathrm{d}t$, then $K(x) = \displaystyle\int \frac{e^{-x}\ln(1+e^x)}{(1+2e^{-x})^2}\,\mathrm{d}x$.

(c) Calculate $M(\theta) = \displaystyle\int \frac{\mathrm{d}\theta}{(\cos\theta + \sin\theta)(2\cos\theta + \sin\theta)}$. (Set $t = \tan\theta$ and use the question $2°(a)$.)

Exercise 2 **(10 points)**

Let

$$f(x) = \frac{\ln\left(\dfrac{e^x + \cos(2x) - 2}{x}\right) + \sin x + \sqrt{1 + ax} - b}{x^2}.$$

Calculate a and b so that $\lim\limits_{x\to 0} f(x)$ is finite. What is then this limit ?

Exercise 3 **(10 points)**

Given a function f defined by

$$f(x) = x\left[\sqrt[3]{1 + \sin\frac{3}{x}} + e^{-1+\cos\frac{1}{x}} + \ln\left(\frac{2+x}{3+x}\right)\right].$$

Determine the equation of the asymptote (D) to the representative curve (C) of f at $+\infty$.

Precise the relative position of (C) and (D) at $+\infty$.

Exercise 4 ______________________________ **(8 points)**

Calculate the double integral

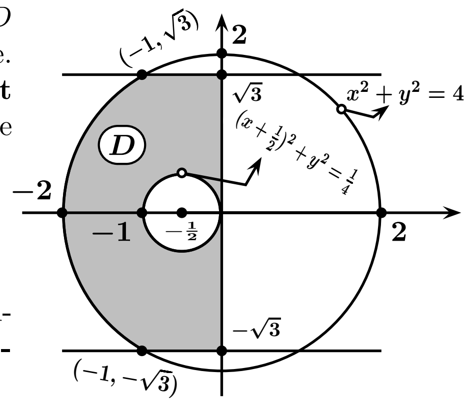
$$I = \int_0^\pi \mathrm{d}y \int_y^\pi \cos y \, e^{\cos x} \, \mathrm{d}x.$$

Exercise 5 ______________________________ **(16 points)**

1° Consider the shaded domain D shown in the adjacent figure. Write down in **two different forms**, the expression of the double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

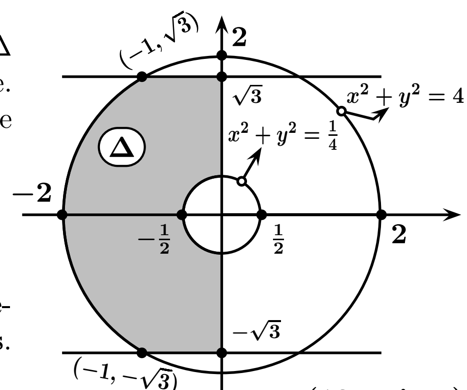
as a succession of two simple integrals, using **cartesian coordinates**.



2° Consider the shaded domain Δ shown in the adjacent figure. Write down the expression of the double integral

$$I = \iint_\Delta f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

as a succession of two simple integrals, using **polar coordinates**.



Exercise 6 ______________________________ **(12 points)**

1° (a) Calculate $\displaystyle\int \frac{\mathrm{d}z}{z\sqrt{z^2 - z}}$ by setting $t = \dfrac{1}{z}$.

(b) Solve the following differential equation :

$$x^2 y' = y\sqrt{y^2 - yx} + xy, \qquad (x > 0).$$

2° Solve the following differential equation :

$$(\cos^2 x \tan x)\; y' - y = 1.$$

Exercise 1

1° (a) We have

$$\begin{aligned} I(x) &= \frac{-1}{2}\int \frac{(-2x+2)-2}{2-x^2+2x}\,\mathrm{d}x \\ &= \frac{-1}{2}\int \frac{(-2x+2)\,\mathrm{d}x}{2-x^2+2x} + \int \frac{\mathrm{d}x}{-x^2+2x+2} = J_1(x) + J_2(x) \end{aligned}$$

where

- $J_1(x) = -\frac{1}{2}\int \frac{(-2x+2)\,\mathrm{d}x}{2-x^2+2x} = -\frac{1}{2}\ln\left|2-x^2+2x\right| + \mathbf{c}_1.$
- $J_2(x) = \int \frac{\mathrm{d}(x-1)}{3-(x-1)^2} = \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+x-1}{\sqrt{3}-x+1}\right| + \mathbf{c}_2.$

(b) Calculation of $J(t) = \int \frac{\mathrm{d}t}{1+t+2\sqrt{1-t}}$. Let $x = \sqrt{1-t}$, then $t = 1-x^2$ and $\mathrm{d}t = -2x\,\mathrm{d}x$:

$$J(t) = -2\int \frac{x\,\mathrm{d}x}{2-x^2+2x} = -2I(x).$$

Thus : $J(t) = \ln(1+t+2\sqrt{1-t}) - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{1-t}-1}{\sqrt{3}-\sqrt{1-t}+1}\right| + \text{cnst}.$

2° (a)
- We have $\frac{1}{(1+t)(2+t)} = \frac{1}{1+t} - \frac{1}{2+t}.$
- Then $I(t) = \int \frac{\mathrm{d}t}{(1+t)(2+t)} = \ln|1+t| - \ln|2+t| + \text{cnst}.$

(b) Calculation of $J(t) = \int \frac{\ln(1+t)}{(2+t)^2}\,\mathrm{d}t$. Integrate by parts, with

$$\begin{cases} u = \ln(1+t) \\ \mathrm{d}v = \dfrac{\mathrm{d}t}{(2+t)^2} \end{cases} \Rightarrow \begin{cases} \mathrm{d}u = \dfrac{\mathrm{d}t}{1+t}; \\ v = \dfrac{-1}{2+t}. \end{cases}$$

Hence

$$\begin{aligned} J(t) &= -\frac{\ln(1+t)}{2+t} + \int \frac{\mathrm{d}t}{(1+t)(2+t)} \\ &= -\frac{\ln(1+t)}{2+t} + \ln|1+t| - \ln|2+t| + \text{cnst}. \end{aligned}$$

Calculation of $K(x) = \displaystyle\int \frac{e^{-x}\ln(1+e^x)}{(1+2e^{-x})^2}\,\mathrm{d}x$. Set $t = e^x$, then $\mathrm{d}x = \dfrac{\mathrm{d}t}{t}$, it yields :

$$K(x) = \int \frac{\frac{1}{t}\ln(1+t)}{(1+2\frac{1}{t})^2}\frac{\mathrm{d}t}{t} = \int \frac{\ln(1+t)}{(2+t)^2}\,\mathrm{d}t = J(t).$$

(c) Calculation of $M(\theta) = \displaystyle\int \frac{\mathrm{d}\theta}{(\cos\theta+\sin\theta)(2\cos\theta+\sin\theta)}$. We have :

$$\begin{aligned} M(\theta) &= \int \frac{1}{(1+\tan\theta)(2+\tan\theta)}\frac{\mathrm{d}\theta}{\cos^2\theta} \\ &\underset{t=\tan\theta}{=} \int \frac{\mathrm{d}t}{(1+t)(2+t)} = I(t). \end{aligned}$$

Thus : $M(\theta) = \ln|1+\tan\theta| - \ln|2+\tan\theta| + \text{cnst}.$

Exercise 2

In neighborhood of $x = 0$ and to order 3 :

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\varepsilon(x) \qquad \text{and} \qquad \cos(2x) = 1 - 2x^2 + x^3\varepsilon(x).$$

Hence, in neighborhood of $x = 0$ and to order 2 :

$$\begin{aligned} \ln\left(\frac{e^x+\cos(2x)-2}{x}\right) &= \ln\left(1 - \frac{3x}{2} + \frac{x^2}{6}\right) + x^2\varepsilon(x) \\ &= -\frac{3x}{2} + \frac{x^2}{6} - \frac{1}{2}\left(-\frac{3x}{2} + \frac{x^2}{6}\right)^2 + x^2\varepsilon(x) \\ &= -\frac{3x}{2} - \frac{23x^2}{24} + x^2\varepsilon(x). \end{aligned}$$

In neighborhood of $x = 0$ and to order 2 :

$$\sin(x) = x + x^2\varepsilon(x) \qquad \text{and} \qquad \sqrt{1+ax} = 1 + \frac{ax}{2} - \frac{a^2x^2}{8} + x^2\varepsilon(x).$$

$$\begin{aligned} \text{Therefore : } f(x) &= \frac{-\frac{3x}{2} - \frac{23x^2}{24} + x + 1 + \frac{ax}{2} - \frac{a^2x^2}{8} - b}{x^2} + \varepsilon(x) \\ &= \frac{1-b}{x^2} + \frac{a-1}{2x} - \frac{23}{24} - \frac{a^2}{8} + \varepsilon(x). \end{aligned}$$

Hence $\lim\limits_{x\to 0} f(x)$ is finite if and only if $b = 1$ and $a = 1$. In this case : $\lim\limits_{x\to 0} f(x) = -\dfrac{23}{24} - \dfrac{a^2}{8} = -\dfrac{13}{12}$.

Exercise 3

Let $x = \dfrac{1}{t}$ and give the f.e. of $tf\left(\dfrac{1}{t}\right)$ to order 2 in neighborhood of $t=0$:

$$tf\left(\frac{1}{t}\right) = \sqrt[3]{1+\sin(3t)} + e^{-1+\cos(t)} + \ln\frac{1+2t}{1+3t}.$$

In neighborhood of $t=0$ and to order 2 :

- $$\begin{aligned}\sqrt[3]{1+\sin(3t)} &= \sqrt[3]{1+3t} + t^2\varepsilon(t)\\ &= 1+\frac{1}{3}(3t) - \frac{1}{9}(3t)^2 + t^2\varepsilon(t) = 1+t-t^2+t^2\varepsilon(t).\end{aligned}$$
- $e^{-1+\cos(t)} = e^{-t^2/2} + t^2\varepsilon(t) = 1 - \dfrac{t^2}{2} + t^2\varepsilon(t).$
- $$\begin{aligned}\ln\frac{1+2t}{1+3t} &= \ln(1+2t) - \ln(1+3t)\\ &= 2t - \frac{1}{2}(2t)^2 - \left[3t - \frac{1}{2}(3t)^2\right] + t^2\varepsilon(t) = -t + \frac{5t^2}{2} + t^2\varepsilon(t).\end{aligned}$$

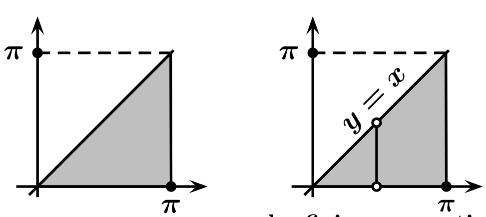
From which it follows that

$$tf\left(\frac{1}{t}\right) = 2 + t^2 + t^2\varepsilon(t) \Leftrightarrow f(x) = 2x + \frac{1}{x} + \frac{1}{x}\varepsilon(x).$$

The equation of (D) is $y = 2x$. Since $f(x) - y \underset{+\infty}{\simeq} \dfrac{1}{x} > 0$ then the curve (C) is above (D) at $+\infty$.

Exercise 4

To evaluate the double integral $I = \displaystyle\int_0^\pi \mathrm{d}y \int_y^\pi \cos(y)e^{\cos x}\,\mathrm{d}x$, we should change the order of integration. The domain (D) of integration of I is bounded by $y=0$, $y=\pi$ as well as by $x=y$ (to the left) and $y=\pi$ (to the right) :



by fixing x : vertically

Changing the order of integration (vertically) : $D : \begin{cases} 0 \le x \le \pi; \\ 0 \le y \le x. \end{cases}$

Thus we get

$$\begin{aligned} I &= \int_0^\pi \mathrm{d}x \int_0^x \cos(y)e^{\cos x}\,\mathrm{d}y = \int_0^\pi e^{\cos(x)}\big[\sin(y)\big]_0^x\,\mathrm{d}x\\ &= \int_0^\pi \sin x e^{\cos x}\,\mathrm{d}x = \big[-e^{\cos x}\big]_0^\pi = -e^{-1} + e.\end{aligned}$$

Exercise 5

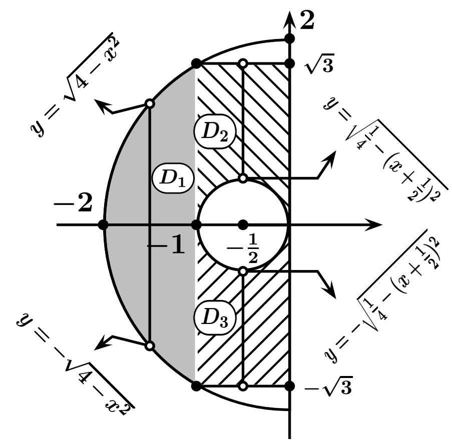
1° **First method : Vertically (by fixing x).**

$D = D_1 \cup D_2 \cup D_3$ with :

$D_1 : \begin{cases} -2 \le x \le -1 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{cases}$

$D_2 : \begin{cases} -1 \le x \le 0 \\ \sqrt{\frac{1}{4} - (x+\frac{1}{2})^2} \le y \le \sqrt{3} \end{cases}$

$D_3 : \begin{cases} -1 \le x \le 0 \\ -\sqrt{3} \le y \le -\sqrt{\frac{1}{4} - (x+\frac{1}{2})^2} \end{cases}$



Hence $I = I_1 + I_2 + I_3$ with

$$I_1 = \int_{-2}^{-1} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y, \qquad I_2 = \int_{-1}^{0} \mathrm{d}x \int_{\sqrt{\frac{1}{4}-(x+\frac{1}{2})^2}}^{\sqrt{3}} f(x,y)\,\mathrm{d}y,$$

$$I_3 = \int_{-1}^{0} \mathrm{d}x \int_{-\sqrt{3}}^{-\sqrt{\frac{1}{4}-(x+\frac{1}{2})^2}} f(x,y)\,\mathrm{d}y.$$

Second method : Horizontally (by fixing y).

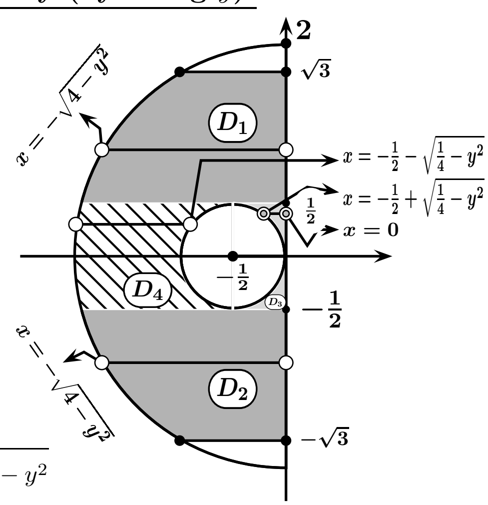
$D = D_1 \cup D_2 \cup D_3 \cup D_4$ with :

$D_1 : \begin{cases} \frac{1}{2} \le y \le \sqrt{3} \\ -\sqrt{4-y^2} \le x \le 0 \end{cases}$

$D_2 : \begin{cases} -\sqrt{3} \le y \le -\frac{1}{2} \\ -\sqrt{4-y^2} \le x \le 0 \end{cases}$

$D_3 : \begin{cases} -\frac{1}{2} \le y \le \frac{1}{2} \\ -\frac{1}{2} + \sqrt{\frac{1}{4} - y^2} \le x \le 0 \end{cases}$

$D_4 : \begin{cases} -\frac{1}{2} \le y \le \frac{1}{2} \\ -\sqrt{4-y^2} \le x \le -\frac{1}{2} - \sqrt{\frac{1}{4} - y^2} \end{cases}$



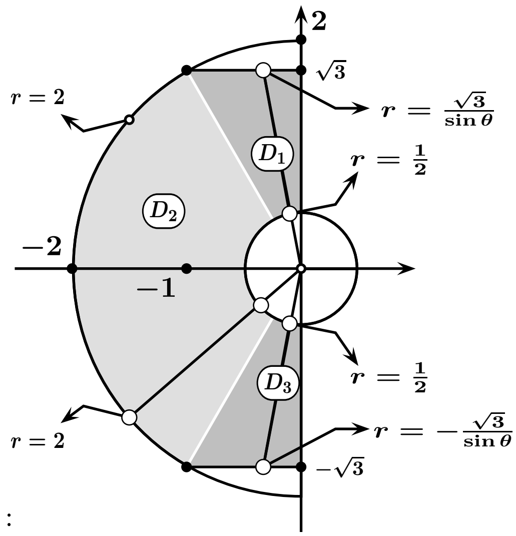
Hence $I = J_1 + J_2 + J_3 + J_4$ with

$$J_1 = \int_{-\sqrt{3}}^{-\frac{1}{2}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{0} f(x,y)\,\mathrm{d}x \qquad J_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{-\frac{1}{2}-\sqrt{\frac{1}{4}-y^2}} f(x,y)\,\mathrm{d}x$$

$$J_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y \int_{-\frac{1}{2}+\sqrt{\frac{1}{4}-y^2}}^{0} f(x,y)\,\mathrm{d}x, \qquad J_4 = \int_{\frac{1}{2}}^{\sqrt{3}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{0} f(x,y)\,\mathrm{d}x.$$

2° **In polar coordinates :**

$\Delta = D_1 \cup D_2 \cup D_3$ where :



$$D_1 : \begin{cases} \dfrac{\pi}{2} \leq \theta \leq \dfrac{2\pi}{3} \\ \dfrac{1}{2} \leq r \leq \dfrac{\sqrt{3}}{\sin\theta} \end{cases}$$

$$D_2 : \begin{cases} \dfrac{2\pi}{3} \leq \theta \leq \dfrac{4\pi}{3} \\ \dfrac{1}{2} \leq r \leq 2 \end{cases}$$

$$D_3 : \begin{cases} \dfrac{4\pi}{3} \leq \theta \leq \dfrac{3\pi}{2} \\ \dfrac{1}{2} \leq r \leq -\dfrac{\sqrt{3}}{\sin\theta} \end{cases}$$

Hence $I = I_1 + I_2 + I_3$ with :

$$I_1 = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \mathrm{d}\theta \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{\sin\theta}} f(r\cos\theta, \sin\theta) r \,\mathrm{d}r, \quad I_2 = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \mathrm{d}\theta \int_{\frac{1}{2}}^{2} f(r\cos\theta, \sin\theta) r \,\mathrm{d}r$$

$$I_3 = \int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} \mathrm{d}\theta \int_{\frac{1}{2}}^{\frac{-\sqrt{3}}{\sin\theta}} f(r\cos\theta, \sin\theta) r \,\mathrm{d}r.$$

Exercise 6

1° (a) Let $t = \dfrac{1}{z}$, then $z = \dfrac{1}{t}$ and $\mathrm{d}z = -\dfrac{\mathrm{d}t}{t^2}$, it yields :

$$I(z) = \int \frac{\mathrm{d}z}{z\sqrt{z^2 - z}} = \int \frac{1}{\frac{1}{t}\sqrt{\frac{1}{t^2} - \frac{1}{t}}} \frac{-\,\mathrm{d}t}{t^2}.$$

Then $I(z) = -\displaystyle\int \frac{\mathrm{d}t}{\sqrt{1-t}} = 2\sqrt{1-t} + \text{cnst} = 2\sqrt{1 - \frac{1}{z}} + \text{cnst}.$

(b) We may write

$$y' = \frac{y}{x}\sqrt{\frac{y^2}{x^2} - \frac{y}{x}} + \frac{y}{x}.$$

It presents a homogeneous differential equation. Let $z = \dfrac{y}{x}$, then $y' = xz' + z$. It gives :

$$xz' + z = z\sqrt{z^2 - z} + z \Leftrightarrow \frac{\mathrm{d}z}{z\sqrt{z^2 - z}} = \frac{\mathrm{d}x}{x}.$$

Hence, by integration and due to result of 1°-(a) we get :

$$2\sqrt{1 - \frac{x}{y}} = \ln|x| + \text{cnst}.$$

2° **First method :** It is a differential equation with separable variables :

$$y'\cos^2 x\tan x = 1+y \quad\Leftrightarrow\quad \frac{\mathrm{d}y}{y+1} = \frac{1}{\tan x}\times\frac{\mathrm{d}x}{\cos^2 x}$$
$$\Leftrightarrow\quad \frac{\mathrm{d}y}{y+1} = \frac{\mathrm{d}(\tan x)}{\tan x}.$$

By integration we get :

$$\ln|1+y| = \ln|\tan x| + \text{cnst}.$$

Second method : We write :

$$y' - \frac{1}{\cos^2 x\tan x}y = \frac{1}{\cos^2 x\tan x} \tag{1}$$

It represents a linear diff. equation. Seeking for the general solution of this equation in the form $y = uv$. Then $y' = u'v + v'u$. It yields

$$u'v + \left[v' - \frac{1}{\cos^2 x\tan x}v\right]u = \frac{1}{\cos^2 x\tan x} \tag{2}$$

1^st^ step : Calculation of v. We choose v such that

$$v' - \frac{1}{\cos^2 x\tan x}v = 0 \Leftrightarrow \frac{\mathrm{d}v}{v} = \frac{1}{\tan x}\times\frac{\mathrm{d}x}{\cos^2 x}.$$

By direct integration :

$$\ln v = \ln\tan x, \quad \text{hence} \quad v = \tan x.$$

2^nd^ step : Calculation of u.

$$u'v = \frac{1}{\cos^2 x\tan x} \Leftrightarrow u' = \frac{1}{\cos^2 x\tan^2 x}.$$

By integration, we get :

$$u = \int \frac{1}{\tan^2 x}\times\frac{\mathrm{d}x}{\cos^2 x} = \int \frac{\mathrm{d}(\tan x)}{\tan^2 x} = -\frac{1}{\tan x} + k.$$

Finally :

$$y = \left(-\frac{1}{\tan x} + k\right)\tan x,$$

where k is an arbitrary constant in $\mathbb{R}$.

LEBANESE UNIVERSITY Date: 12 February 2016
Faculty Of Sciences (I)

Final exam **M1109** **Duration: 2 hours**

ANSWER PAGE 136

Exercise 1 **(12 points)**
Let
$$f(x)=\frac{\tan x+\sqrt[3]{1+3x+ax^2}-e^{(2x+x^2)}}{x^2(1-e^x)}.$$
Calculate a so that $\lim_{x\to 0} f(x)$ is finite. What is then this limit ?

Exercise 2 **(8 points)**
Let f be the function defined by
$$f(x)=x\ln\left(\sqrt{x^2+2x+2}-x\right).$$
Determine the equation of the asymptote (D) to the representative curve (C) of the function f at $+\infty$. Precise the relative position of (C) with respect to (D) at $+\infty$.

Exercise 3 **(14 points)**

1° (a) Calculate $I(x)=\displaystyle\int \frac{1}{x(1+x^2)}\,\mathrm{d}x$ and $J(x)=\displaystyle\int \frac{1}{x^2}\arctan(x)\,\mathrm{d}x$.

(b) Solve the following differential equation :
$$x(1+x)y'-y=\arctan x.$$

2° Solve the following differential equation :
$$\left(e^{\frac{y}{x}}-\cos\left(\frac{y}{x}\right)\right)y'=\left(\frac{y}{x}-1\right)e^{\frac{y}{x}}-\frac{y}{x}\cos\left(\frac{y}{x}\right)+\sin\left(\frac{y}{x}\right).$$

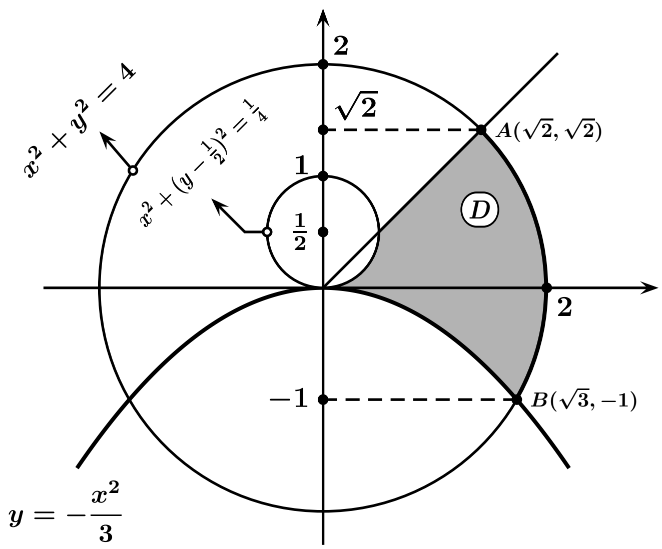
Exercise 4 **(16 points)**
Calculate the following integrals
$$I(x)=\int\frac{x+1}{\sqrt{-4x^2+4x+3}}\,\mathrm{d}x,\quad J(x)=\int\frac{2-\tan x}{(\cos x+2\sin x)\cos x}\,\mathrm{d}x,$$
$$K(x)=\int\frac{\mathrm{d}x}{(2+\cos x)\tan\frac{x}{2}}.$$

Exercise 5 ____ **(20 points)**

Consider the following double integral

$$I = \iint_D f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

where D is the shaded domain in the figure below.



1° Using **cartesian coordinates**, write down I by 2 ways, as succession of two simple integrals.

2° Using **polar coordinates**, write down I, as succession of two simple integrals.

Exercise 1

There must be simplified by x^3 because the f.e. of $x^2(1-e^x)$ begins with a term of x^3. To find the limit of $f(x)$ we must give the f.e. of each usual function to order 3.

- $x^2(1-e^x) = x^2[1-(1+x)] + x^3\varepsilon(x) = -x^3 + x^3\varepsilon(x)$.
- $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{x - \dfrac{x^3}{6}}{1-\dfrac{x^2}{2}} + x^3\varepsilon(x) = x + \dfrac{x^3}{3} + x^3\varepsilon(x)$.
- $$\begin{aligned}\sqrt[3]{1+3x+ax^2} &= [1+(3x+ax^2)]^{\frac{1}{3}}\\ &= 1+\frac{1}{3}(3x+ax^2) - \frac{1}{9}(3x+ax^2)^2 + \frac{5}{81}(3x)^3 + x^3\varepsilon(x)\\ &= 1+x+\left(\frac{a}{3}-1\right)x^2 + \frac{1}{3}(5-2a)x^3 + x^3\varepsilon(x).\end{aligned}$$
- $$\begin{aligned}e^{(2x+x^2)} &= 1+2x+x^2+\frac{1}{2}(2x+x^2)^2+\frac{1}{6}(2x)^3 + x^3\varepsilon(x)\\ &= 1+2x+3x^2+\frac{10}{3}x^3 + x^3\varepsilon(x).\end{aligned}$$

Therefore : $f(x) = \dfrac{12-a}{3x} + \dfrac{4+2a}{3} + \varepsilon(x)$.

Then $\lim\limits_{x\to 0} f(x)$ exists iff $a = 12$. In this case : $\lim\limits_{x\to 0} f(x) = \dfrac{4+2a}{3} = \dfrac{28}{3}$.

Exercise 2

Set $x = \dfrac{1}{t}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. We have then

$$f\left(\frac{1}{t}\right) = \frac{1}{t}\ln\left(\sqrt{\frac{1}{t^2}+\frac{2}{t}+2} - \frac{1}{t}\right) \Rightarrow tf\left(\frac{1}{t}\right) = \ln\frac{\sqrt{1+2t+2t^2}-1}{t}.$$

Let us give the f.e. of $tf\left(\dfrac{1}{t}\right)$ to order 2 in neighborhood of $t=0$.

In neighborhood of $t=0$ and to order 3 we have :

$$\begin{aligned}\sqrt{1+2t+2t^2} &= [1+(2t+2t^2)]\\ &= 1+\frac{1}{2}(2t+2t^2) - \frac{1}{8}(2t+2t^2)^2 + \frac{1}{16}(2t)^3 + t^3\varepsilon(t)\\ &= 1+t+\frac{t^2}{2} - \frac{t^3}{2} + t^3\varepsilon(t).\end{aligned}$$

Thus : $\ln \dfrac{\sqrt{1+2t+2t^2}-1}{t} = \ln\left(1+\left(\dfrac{t}{2}-\dfrac{t^2}{2}\right)\right)+t^2\varepsilon(t)$

$$= \frac{t}{2}-\frac{t^2}{2}-\frac{1}{2}\left(\frac{t}{2}-\frac{t^2}{2}\right)^2+t^2\varepsilon(t)$$

$$= \frac{t}{2}-\frac{5t^2}{8}+t^2\varepsilon(t).$$

Then

$$tf\left(\frac{1}{t}\right)=\frac{t}{2}-\frac{5t^2}{8}+t^2\varepsilon(t).$$

Returning back in x, we obtain :

$$f(x)=\frac{1}{2}-\frac{5}{8x}+\frac{1}{x}\varepsilon(x).$$

So the equation of the asymptote is $y_a=\dfrac{1}{2}$ and is above the curve representing f since $f(x)-y_a \underset{+\infty}{\simeq} -\dfrac{5}{8x}<0$.

Exercise 3

1° (a) $$I(x) = \int \frac{1}{x(1+x^2)}\,\mathrm{d}x = \int\frac{\mathrm{d}x}{x}-\int\frac{x\,\mathrm{d}x}{1+x^2} = \ln|x|-\frac{1}{2}\ln(1+x^2)+\text{cnst}.$$

Calculate $J(x)=\displaystyle\int\frac{1}{x^2}\arctan(x)\,\mathrm{d}x$ by integration by parts with

$$u=\arctan x \quad\Rightarrow \mathrm{d}u=\frac{\mathrm{d}x}{1+x^2}$$
$$\mathrm{d}v=\frac{\mathrm{d}x}{x^2} \quad\Rightarrow v=-\frac{1}{x}.$$

We get then

$$J(x)=-\frac{1}{x}\arctan(x)+\int\frac{1}{x(1+x^2)}\,\mathrm{d}x=-\frac{1}{x}\arctan(x)+I(x).$$

Hence :

$$J(x)=-\frac{1}{x}\arctan(x)+\ln|x|-\frac{1}{2}\ln(1+x^2)+\text{cnst}.$$

(b) This is a linear differential equation that is written in the form

$$y' - \frac{1}{x(1+x)}y = \frac{1}{x(1+x)}\arctan(x) \tag{1}$$

Search y in the form $y = uv$, then $y' = u'v + v'u$. We find by replacing y and y' in (1) :

$$u'v + \left[v' - \frac{1}{x(1+x)}v\right]u = \frac{1}{x(1+x)}\arctan(x) \tag{2}$$

Calculation of v : We choose v such that :

$$v' - \frac{1}{x(1+x)}v = 0 \Rightarrow \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}x}{x(1+x)}.$$

By integration, we get

$$\ln v = \int \frac{1}{x(1+x)}\,\mathrm{d}x = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x.$$

So $\ln v = \ln \dfrac{x}{1+x}$ and then $v = \dfrac{x}{1+x}$.

Calculation of u : From (2), we have

$$u'v = \frac{1}{x(1+x)}\arctan(x) \Rightarrow u' = \frac{1}{x^2}\arctan x.$$

According to the results of the question $1°(a)$, we obtain

$$u(x) = J(x) = -\frac{1}{x}\arctan(x) + \ln|x| - \frac{1}{2}\ln(1+x^2) + \text{cnst}.$$

Finally, $y = uv = \left(-\dfrac{\arctan x}{x} + \ln|x| - \dfrac{1}{2}\ln(1+x^2) + C\right)\dfrac{x}{1+x}$.

2° It is HDE since $y' = \dfrac{\left(\frac{y}{x} - 1\right)e^{\frac{y}{x}} - \frac{y}{x}\cos\left(\frac{y}{x}\right) + \sin\left(\frac{y}{x}\right)}{e^{\frac{y}{x}} - \cos\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$.

Set $z = \dfrac{y}{x}$, then $y = xz$ and $y' = xz' + z$. Hence

$$xz' + z = \frac{(z-1)e^z - z\cos(z) + \sin(z)}{e^z - \cos(z)} \Rightarrow xz' = \frac{\sin(z) - e^z}{e^z - \cos(z)}.$$

Therefore

$$\int \frac{e^z - \cos(z)}{\sin(z) - e^z}\,\mathrm{d}z = \int \frac{\mathrm{d}x}{x}.$$

Thus : $-\ln|\sin(z) - e^z| = \ln|x| + C$ where $z = \dfrac{y}{x}$.

Exercise 4

- **Calculation of** $\boldsymbol{I(x) = \int \frac{x+1}{\sqrt{-4x^2+4x+3}}\,\mathrm{d}x}$ **:**

$$\begin{aligned} I(x) &= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{-4x^2+4x+3}}\,\mathrm{d}x \\ &= -\frac{1}{8}\int \frac{-8x+4}{\sqrt{-4x^2+4x+3}}\,\mathrm{d}x + \frac{3}{4}\int \frac{\mathrm{d}(2x-1)}{\sqrt{2^2-(2x-1)^2}} \\ &= -\frac{1}{4}\sqrt{-4x^2+4x+3} + \frac{3}{4}\arcsin\frac{2x-1}{2} + \text{cnst.} \end{aligned}$$

- **Calculation of** $\boldsymbol{J(x) = \int \frac{2-\tan x}{(\cos x + 2\sin x)\cos x}\,\mathrm{d}x}$ **:** We write :

$$J(x) = \int \frac{2-\tan x}{1+2\tan x} \times \frac{\mathrm{d}x}{\cos^2 x}.$$

Set $t = \tan x$, then $\mathrm{d}t = \dfrac{\mathrm{d}x}{\cos^2 x}$, and :

$$J(x) = \int \frac{2-t}{1+2t}\,\mathrm{d}t = -\frac{1}{2}\int \frac{2t-4}{2t+1}\,\mathrm{d}t = -\frac{1}{2}\int \mathrm{d}t + \frac{5}{2}\int \frac{\mathrm{d}t}{2t+1}.$$

Therefore : $J(x) = -\dfrac{1}{2}t + \dfrac{5}{4}\ln|1+2t| + \text{cnst}$, where $t = \tan x$.

- **Calculation of** $\boldsymbol{K(x) = \int \frac{\mathrm{d}x}{(2+\cos x)\tan\frac{x}{2}}}$ **:**

Set $t = \tan\dfrac{x}{2}$, then $\dfrac{x}{2} = \arctan(t)$ and $\mathrm{d}x = 2\dfrac{\mathrm{d}t}{1+t^2}$. We get then

$$\begin{aligned} K(x) &= 2\int \frac{1}{\left(2+\dfrac{1-t^2}{1+t^2}\right)t} \times \frac{\mathrm{d}t}{1+t^2} \\ &= 2\int \frac{1}{(3+t^2)t}\,\mathrm{d}t = \frac{2}{3}\int \frac{3+t^2-t^2}{t(3+t^2)}\,\mathrm{d}t \\ &= \frac{2}{3}\int \frac{\mathrm{d}t}{t} - \frac{2}{3}\int \frac{t}{3+t^2}\,\mathrm{d}t. \end{aligned}$$

Thus : $K(x) = \dfrac{2}{3}\ln|t| - \dfrac{1}{3}\ln(3+t^2) + \text{cnst}, \quad \text{where} \quad t = \tan\dfrac{x}{2}.$

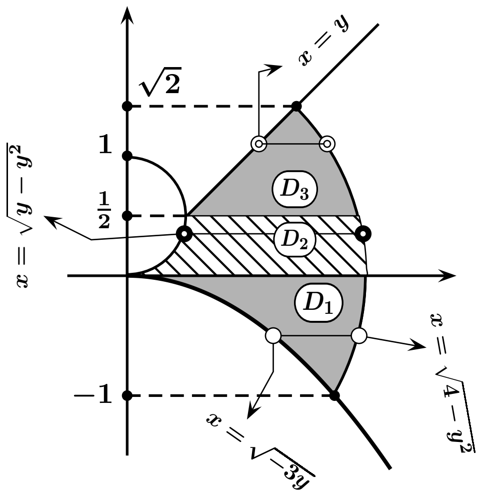
Exercise 5

Horizontally : We have

$$D = D_1 \cup D_2 \cup D_3$$

where

- $D_1 : \begin{cases} -1 \leq y \leq 0; \\ \sqrt{-3y} \leq x \leq \sqrt{4-y^2}. \end{cases}$
- $D_2 : \begin{cases} 0 \leq y \leq \frac{1}{2}; \\ \sqrt{y-y^2} \leq x \leq \sqrt{4-y^2}. \end{cases}$
- $D_3 : \begin{cases} \frac{1}{2} \leq y \leq \sqrt{2}; \\ y \leq x \leq \sqrt{4-y^2}. \end{cases}$

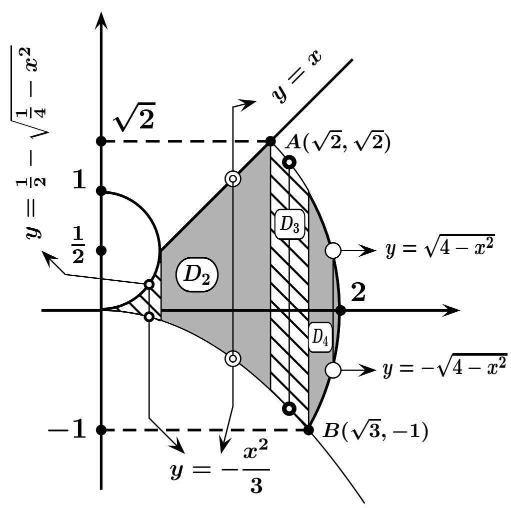


Thus :

$$I = \int_{-1}^{0} \mathrm{d}y \int_{\sqrt{-3y}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{0}^{\frac{1}{2}} \mathrm{d}y \int_{\sqrt{y-y^2}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{\frac{1}{2}}^{\sqrt{2}} \mathrm{d}y \int_{y}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x.$$

Vertically : $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where

- $D_1 : \begin{cases} 0 \leq x \leq \frac{1}{2}; \\ -\frac{x^2}{3} \leq y \leq \frac{1}{2} - \sqrt{\frac{1}{4} - x^2}. \end{cases}$
- $D_2 : \begin{cases} \frac{1}{2} \leq x \leq \sqrt{2}; \\ -\frac{x^2}{3} \leq y \leq x. \end{cases}$
- $D_3 : \begin{cases} \sqrt{2} \leq x \leq \sqrt{3}; \\ -\frac{x^2}{3} \leq y \leq \sqrt{4-x^2}. \end{cases}$
- $D_4 : \begin{cases} \sqrt{3} \leq x \leq 2; \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}. \end{cases}$

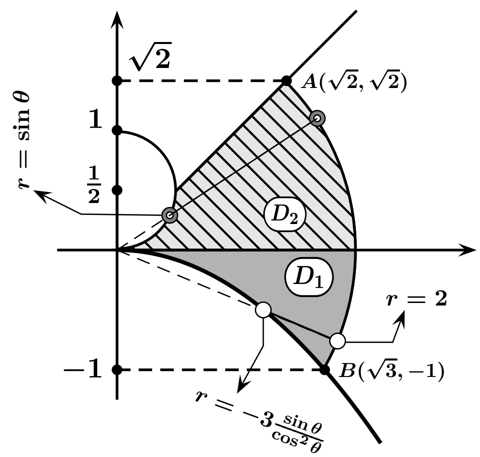


Thus :

$$I = \int_{0}^{\frac{1}{2}} \mathrm{d}x \int_{-\frac{x^2}{3}}^{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}} f(x,y)\,\mathrm{d}y + \int_{\frac{1}{2}}^{\sqrt{2}} \mathrm{d}x \int_{-\frac{x^2}{3}}^{x} f(x,y)\,\mathrm{d}y$$
$$+ \int_{\sqrt{2}}^{\sqrt{3}} \mathrm{d}x \int_{-\frac{x^2}{3}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y + \int_{\sqrt{3}}^{2} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y.$$

<u>In polar coordinates :</u> We have

$$D = D_1 \cup D_2$$



where

- $D_1 : \begin{cases} -\dfrac{\pi}{6} \leq \theta \leq 0; \\ -\dfrac{3\sin\theta}{\cos^2\theta} \leq r \leq 2. \end{cases}$

- $D_2 : \begin{cases} 0 \leq \theta \leq \dfrac{\pi}{4}; \\ \sin\theta \leq r \leq 2. \end{cases}$

Thus :

$$I = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_{-\frac{\pi}{6}}^{0} \mathrm{d}\theta \int_{-\frac{3\sin\theta}{\cos^2\theta}}^{2} f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r \\ I_2 &= \int_{0}^{\frac{\pi}{4}} \mathrm{d}\theta \int_{\sin\theta}^{2} f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r. \end{aligned}$$

LEBANESE UNIVERSITY Date: 14 September 2016
Faculty Of Sciences (I)

Final exam **M1109** **Duration: 2 hours**

ANSWER PAGE 144

Exercise 1 **(12 points)**

Let

$$f(x) = \frac{\sin x + \sqrt[3]{\cos(x) + 3x} - e^{(2x+ax^2)}}{x^2(1-e^x)}.$$

Calculate a so that $\lim_{x\to 0} f(x)$ is finite. What is then this limit ?

Exercise 2 **(8 points)**

Let f be the function defined by

$$f(x) = xe^{\frac{\sqrt{x^2+2x+2}}{x}}.$$

Determine the equation of the asymptote (D) of the representative curve (C) of the function f at $+\infty$. Precise the relative position of (C) with respect to (D) at $+\infty$.

Exercise 3 **(14 points)**

1° (a) Calculate $I(x) = \int \frac{1}{x(1+x)}\,\mathrm{d}x$ and $J(x) = \int \frac{1}{x}\ln x\,\mathrm{d}x$.

(b) Solve the following differential equation : $x(1+x)y' - y = x\ln x$.

2° Solve the following differential equation :

$$\left(e^{\frac{y}{x}} - 1\right) y' = \left(\frac{y}{x} - 1\right) e^{\frac{y}{x}}.$$

Exercise 4 **(16 points)**

Calculate the following integrals :

$$I(x) = \int \frac{x}{\sqrt{-4x^2+12x-5}}\,\mathrm{d}x, \qquad J(x) = \int \frac{\sin(2x)}{2+\cos^2 x}\,\mathrm{d}x,$$

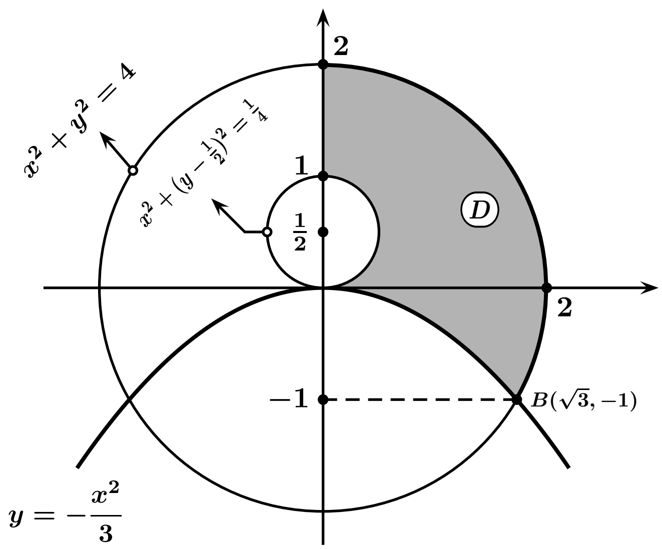
$$K(x) = \int \frac{\tan\frac{x}{2}}{2+\cos x}\,\mathrm{d}x.$$

Exercise 5 **(20 points)**

Consider the following double integral

$$I = \iint_D f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

where D is the shaded domain in the below figure.



1° Using **cartesian coordinates**, write down I by two ways, as succession of two simple integrals.

2° Using **polar coordinates**, write down I, as succession of two simple integrals.

Exercise 1

There must be simplified by x^3 because the f.e. of $x^2(1-e^x)$ begins with a term of x^3. To find the limit of $f(x)$ we must give the f.e. of each usual function to order 3.

- $x^2(1-e^x) = x^2[1-(1+x)] + x^3\varepsilon(x) = -x^3 + x^3\varepsilon(x)$.
- $\sin x = x - \dfrac{x^3}{6} + x^3\varepsilon(x)$.
- $$\begin{aligned}\sqrt[3]{\cos x + 3x} &= \sqrt[3]{1+3x-\frac{x^2}{2}} + x^3\varepsilon(x)\\ &= \left[1+\left(3x-\frac{x^2}{2}\right)\right]^{\frac{1}{3}} + x^3\varepsilon(x)\\ &= 1+\frac{1}{3}\left(3x-\frac{x^2}{2}\right) - \frac{1}{9}\left(3x-\frac{x^2}{2}\right)^2 + \frac{5}{81}(3x)^3 + x^3\varepsilon(x)\\ &= 1+x-\frac{7}{6}x^2+2x^3+x^3\varepsilon(x).\end{aligned}$$
- $$\begin{aligned}e^{(2x+ax^2)} &= 1+2x+ax^2+\frac{1}{2}(2x+ax^2)^2+\frac{1}{6}(2x)^3+x^3\varepsilon(x)\\ &= 1+2x+(a+2)x^2+\left(2a+\frac{4}{3}\right)x^3+x^3\varepsilon(x).\end{aligned}$$

Therefore : $f(x) = \dfrac{\frac{19}{6}+a}{x} + 2a - \dfrac{2}{3} + \varepsilon(x)$.

Then $\lim\limits_{x\to 0} f(x)$ exists iff $a = -\dfrac{19}{6}$.

In this case : $\lim\limits_{x\to 0} f(x) = 2a - \dfrac{2}{3} = -7$.

Exercise 2

Let $x = \dfrac{1}{t}$ $(x \to +\infty \Leftrightarrow t \to 0_+)$. We have then

$$f\left(\frac{1}{t}\right) = \frac{1}{t}e^{t\sqrt{\frac{1}{t^2}+\frac{2}{t}+2}} \Rightarrow tf\left(\frac{1}{t}\right) = e^{\sqrt{1+2t+2t^2}}.$$

Let's giving the f.e. of $tf\left(\dfrac{1}{t}\right)$ to order 2 in neighborhood of $t=0$.

In neighborhood of $t = 0$ and to order 2 we have :

$$\begin{aligned}\sqrt{1+2t+2t^2} &= \left[1+(2t+2t^2)\right]^{\frac{1}{2}}\\ &= 1+\frac{1}{2}(2t+2t^2)-\frac{1}{8}(2t+2t^2)^2+t^2\varepsilon(t)\\ &= 1+t+\frac{t^2}{2}+t^2\varepsilon(t).\end{aligned}$$

Hence :
$$\begin{aligned} tf\left(\frac{1}{t}\right)=e^{\sqrt{1+2t+2t^2}} &= e\times e^{t+\frac{t^2}{2}}+t^2\varepsilon(t)\\ &= e\left[1+\left(t+\frac{t^2}{2}\right)+\frac{1}{2}\left(t+\frac{t^2}{2}\right)^2\right]+t^2\varepsilon(t)\\ &= e\left(1+t+t^2\right)+t^2\varepsilon(t).\end{aligned}$$

Returning back in x, we obtain :

$$f(x)=e\left(1+x+\frac{1}{x}\right)+\frac{1}{x}\varepsilon(x).$$

So the equation of the asymptote is $y_a = e(1+x)$ and is below the curve representing f since $f(x)-y_a \underset{+\infty}{\simeq} \dfrac{e}{x} > 0$.

Exercise 3

1° (a)

- $$\begin{aligned} I(x) &= \int\frac{1}{x(1+x)}\,\mathrm{d}x\\ &= \int\frac{\mathrm{d}x}{x}-\int\frac{\mathrm{d}x}{1+x}\\ &= \ln|x|-\ln|1+x|+\text{cnst}=\ln\left|\frac{x}{1+x}\right|+\text{cnst}.\end{aligned}$$
- $J(x)=\displaystyle\int\ln x\frac{\mathrm{d}x}{x}=\int\ln x\,\mathrm{d}(\ln x)$. Then
$$J(x)=\frac{1}{2}(\ln x)^2+\text{cnst}.$$

(b) This is a linear differential equation that is written in the form

$$y'-\frac{1}{x(1+x)}y=\frac{1}{1+x}\ln x \qquad (1)$$

Search y in the form $y = uv$, then $y' = u'v + v'u$. We find by replacing y and y' in (1) :

$$u'v+\left[v'-\frac{1}{x(1+x)}v\right]u=\frac{1}{1+x}\ln x \qquad (2)$$

Calculation of v : We choose v such that :

$$v' - \frac{1}{x(1+x)}v = 0 \Rightarrow \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}x}{x(1+x)}.$$

By integration, we get

$$\ln v = \int \frac{1}{x(1+x)}\,\mathrm{d}x = \ln\frac{x}{1+x} \Rightarrow v = \frac{x}{1+x}.$$

Calculation of u : From (2), we have

$$u'v = \frac{1}{1+x}\ln x \Rightarrow u' = \frac{1}{x}\ln x.$$

According to the results of the question $1°(a)$, we obtain

$$u(x) = J(x) = \frac{1}{2}(\ln x)^2 + k, \qquad (k \in \mathbb{R}).$$

Finally,

$$y = uv = \left(\frac{1}{2}(\ln x)^2 + k\right)\frac{x}{1+x}.$$

2° It is HDE since

$$y' = \frac{\left(\frac{y}{x} - 1\right)e^{\frac{y}{x}}}{e^{\frac{y}{x}} - 1} = F\left(\frac{y}{x}\right).$$

Set $z = \dfrac{y}{x}$, then $y = xz$ and $y' = xz' + z$. Hence

$$xz' + z = \frac{(z-1)e^z}{e^z - 1} \Rightarrow xz' = \frac{z - e^z}{e^z - 1}.$$

Then :

$$\int \frac{e^z - 1}{z - e^z}\,\mathrm{d}z = \int \frac{\mathrm{d}x}{x}.$$

Hence :

$$-\ln|z - e^z| = \ln|x| + k,$$

where $z = \dfrac{y}{x}$ and k is an arbitrary real constant.

Exercise 4

- **Calculation of $\boldsymbol{I(x) = \int \frac{x}{\sqrt{-4x^2+12x-5}}\,\mathrm{d}x}$:**

$$\begin{aligned} I(x) &= -\frac{1}{8}\int \frac{-8x}{\sqrt{-4x^2+12x-5}}\,\mathrm{d}x \\ &= -\frac{1}{8}\int \frac{-8x+12}{\sqrt{-4x^2+12x-5}}\,\mathrm{d}x + \frac{3}{4}\int \frac{\mathrm{d}(2x-3)}{\sqrt{2^2-(2x-3)^2}} \\ &= -\frac{1}{4}\sqrt{-4x^2+12x-5} + \frac{3}{4}\arcsin\frac{2x-3}{2} + \text{cnst.} \end{aligned}$$

- **Calculation of $\boldsymbol{J(x) = \int \frac{\sin(2x)}{2+\cos^2 x}\,\mathrm{d}x}$.**

First method : We write :

$$J(x) = \int \frac{2\cos x}{2+\cos^2 x}\sin x\,\mathrm{d}x.$$

Set $t = \cos x$, then $\mathrm{d}t = -\sin x\,\mathrm{d}x$, and it follows that :

$$J(x) = -\int \frac{2t}{2+t^2}\,\mathrm{d}t = -\int \frac{\mathrm{d}(2+t^2)}{2+t^2}.$$

So : $J(x) = -\ln(2+t^2) + \text{cnst}$, where $t = \cos x$.

Second method : Set $t = 2+\cos^2 x$, then $\mathrm{d}t = -\sin(2x)\,\mathrm{d}x$. Thus

$$J(x) = -\int \frac{\mathrm{d}t}{t} = -\ln|t| + \text{cte} = -\ln(2+\cos^2 x) + \text{cnst}.$$

- **Calculation of $\boldsymbol{K(x) = \int \frac{\tan\frac{x}{2}\,\mathrm{d}x}{2+\cos x}}$:**

Set $t = \tan\frac{x}{2}$, then $\frac{x}{2} = \arctan(t)$ and $\mathrm{d}x = 2\frac{\mathrm{d}t}{1+t^2}$. We get then

$$K(x) = 2\int \frac{t}{2+\frac{1-t^2}{1+t^2}} \times \frac{\mathrm{d}t}{1+t^2} = 2\int \frac{t}{3+t^2}\,\mathrm{d}t = \int \frac{\mathrm{d}(3+t^2)}{3+t^2}.$$

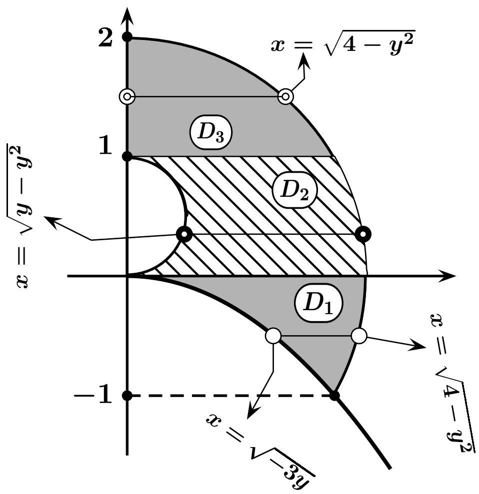
Thus : $K(x) = \ln(3+t^2) + \text{cnst}, \quad$ where $\quad t = \tan\frac{x}{2}$.

Exercise 5

Horizontally : We have

$$D = D_1 \cup D_2 \cup D_3$$

where

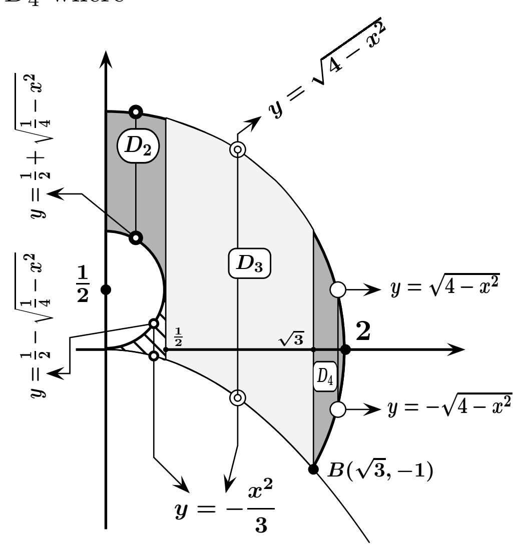


- $D_1 : \begin{cases} -1 \le y \le 0; \\ \sqrt{-3y} \le x \le \sqrt{4-y^2}. \end{cases}$
- $D_2 : \begin{cases} 0 \le y \le 1; \\ \sqrt{y-y^2} \le x \le \sqrt{4-y^2}. \end{cases}$
- $D_3 : \begin{cases} 1 \le y \le 2; \\ 0 \le x \le \sqrt{4-y^2}. \end{cases}$

Then :

$$I = \int_{-1}^{0} \mathrm{d}y \int_{\sqrt{-3y}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{0}^{1} \mathrm{d}y \int_{\sqrt{y-y^2}}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x + \int_{1}^{2} \mathrm{d}y \int_{0}^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x.$$

Vertically : $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where



- $D_1 : \begin{cases} 0 \le x \le \frac{1}{2}; \\ -\frac{x^2}{3} \le y \le \frac{1}{2} - \sqrt{\frac{1}{4} - x^2}. \end{cases}$
- $D_2 : \begin{cases} 0 \le x \le \frac{1}{2}; \\ \frac{1}{2} + \sqrt{\frac{1}{4} - x^2} \le y \le \sqrt{4-x^2}. \end{cases}$
- $D_3 : \begin{cases} \frac{1}{2} \le x \le \sqrt{3}; \\ -\frac{x^2}{3} \le y \le \sqrt{4-x^2}. \end{cases}$
- $D_4 : \begin{cases} \sqrt{3} \le x \le 2; \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}. \end{cases}$

Then :

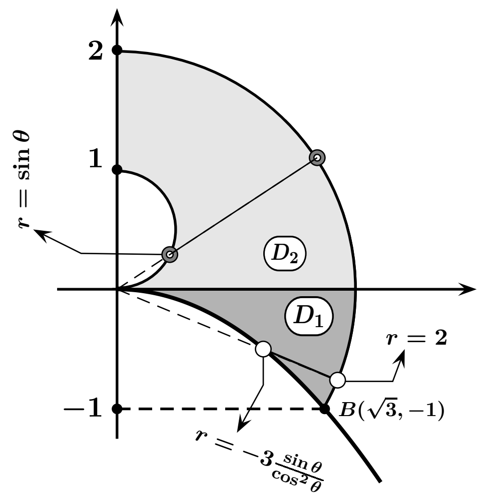
$$I = \int_{0}^{\frac{1}{2}} \mathrm{d}x \int_{-\frac{x^2}{3}}^{\frac{1}{2} - \sqrt{\frac{1}{4} - x^2}} f(x,y)\,\mathrm{d}y + \int_{0}^{\frac{1}{2}} \mathrm{d}x \int_{\frac{1}{2} + \sqrt{\frac{1}{4} - x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y$$

$$+ \int_{\frac{1}{2}}^{\sqrt{3}} \mathrm{d}x \int_{-\frac{x^2}{3}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y + \int_{\sqrt{3}}^{2} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y.$$

By polar coordinates : We have

$$D = D_1 \cup D_2$$

where



- $D_1 : \begin{cases} -\dfrac{\pi}{6} \leq \theta \leq 0; \\ -\dfrac{3\sin\theta}{\cos^2\theta} \leq r \leq 2. \end{cases}$
- $D_2 : \begin{cases} 0 \leq \theta \leq \dfrac{\pi}{2}; \\ \sin\theta \leq r \leq 2. \end{cases}$

Then :

$$I = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_{-\frac{\pi}{6}}^{0} \mathrm{d}\theta \int_{-\frac{3\sin\theta}{\cos^2\theta}}^{2} f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r \\ I_2 &= \int_{0}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\sin\theta}^{2} f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r. \end{aligned}$$

LEBANESE UNIVERSITY Date: 31 January 2017

Faculty Of Sciences (I)

Final exam **M1109** Duration: 2 hours

ANSWER PAGE 152

Exercise 1 **(15 points)**

Let

$$f(x) = \frac{\frac{1}{x^2}\ln(4-3\cos x) + \sqrt[3]{1+\sin(3x)} - \frac{5}{2} - x + ax^2}{x^2(1-e^{2x})}.$$

Determine a so that $\lim_{x\to 0} f(x)$ is finite. What is then this limit ?

Exercise 2 **(15 points)**

1° (a) Verify that : $\dfrac{t^3}{(2+t)^2} = t - 4 + \dfrac{12}{2+t} - \dfrac{8}{(2+t)^2}$.

(b) Calculate $I(x) = \displaystyle\int \frac{\tan^3 x}{(2\cos x + \sin x)^2}\,\mathrm{d}x$.

2° Calculate $J(x) = \displaystyle\int \frac{\mathrm{d}x}{5+3\cos x}$ and $K(x) = \displaystyle\int \frac{\mathrm{d}x}{3-5\cos x}$.

3° Calculate $L(x) = \displaystyle\int \frac{1}{x}\sqrt{\frac{x}{x+1}}\,\mathrm{d}x$.

Exercise 3 **(15 points)**

1° Solve the following differential equation :

$$y' = \frac{xy + (y^2 - xy)\ln\frac{y}{x}}{(xy - x^2)\ln\frac{y}{x}}.$$

2° Calculate $\displaystyle\int e^{-\sqrt{x}}\,\mathrm{d}x$ and then solve the following differential equation :

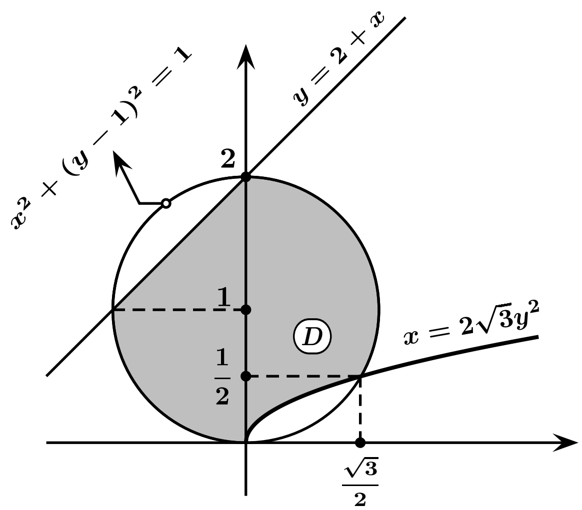
$$2xy' - \sqrt{x}\,y = x.$$

Exercise 4 **(25 points)**

1° Consider the following double integral

$$I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

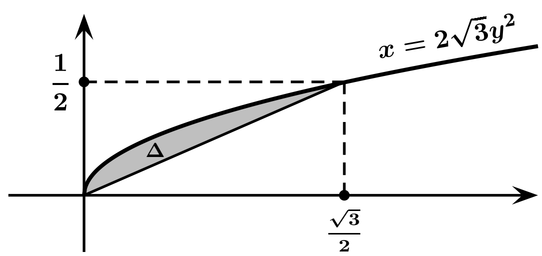
where D is the shaded domain of the figure below.



(a) Write I as a succession of two simple integrals in two different ways, using the **cartesian coordinates**.

(b) Write I as a succession of two simple integrals, using **polar coordinates**.

2° Using polar coordinates, calculate the area of the shaded domain Δ.



Exercise 1

There must be simplification by x^3 because the f.e. of $x^2(1-e^{2x})$ begins with a term in x^3. To find the limit of $f(x)$ we must give the f.e. of each usual function to the order 3.

- $x^2(1-e^{2x}) = x^2[1-(1+2x)] + x^3\varepsilon(x) = -2x^3 + x^3\varepsilon(x)$.
- $$\begin{aligned}\sqrt[3]{1+\sin(3x)} &= \left[1+\left(3x-\frac{(3x)^3}{3!}\right)\right]^{\frac{1}{3}} + x^3\varepsilon(x)\\ &= 1+\frac{1}{3}\left(3x-\frac{9}{2}x^3\right)-\frac{1}{9}\left(3x-\frac{9}{2}x^3\right)^2+\frac{5}{81}(3x)^3+x^3\varepsilon(x)\\ &= 1+x-x^2+\frac{1}{6}x^3+x^3\varepsilon(x).\end{aligned}$$
- To find the f.e. of $\frac{1}{x^2}\ln(4-3\cos x)$ to order 3 we give the f.e. of the function $\ln(4-3\cos x)$ to order 5 :
$$\begin{aligned}\ln(4-3\cos x) &= \ln\left[4-3\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\right]+x^5\varepsilon(x)\\ &= \ln\left[1+\left(\frac{3}{2}x^2-\frac{x^4}{8}\right)\right]+x^5\varepsilon(x)\\ &= \frac{3}{2}x^2-\frac{x^4}{8}-\frac{1}{2}\left(\frac{3}{2}x^2-\frac{x^4}{8}\right)^2+x^5\varepsilon(x)\\ &= \frac{3}{2}x^2-\frac{5}{4}x^4+x^5\varepsilon(x).\end{aligned}$$
Then : $\frac{1}{x^2}\ln(4-3\cos x) = \frac{3}{2}-\frac{5}{4}x^2+x^3\varepsilon(x)$.

Therefore
$$\begin{aligned}f(x) &= \frac{\frac{1}{x^2}\ln(4-3\cos x)+\sqrt[3]{1+\sin(3x)}-\frac{5}{2}-x+ax^2}{x^2(1-e^{2x})}\\ &= \frac{\frac{3}{2}-\frac{5}{4}x^2+1+x-x^2+\frac{1}{6}x^3-\frac{5}{2}-x+ax^2}{-2x^3}+\varepsilon(x)\\ &= \frac{9-4a}{8x}-\frac{1}{12}+\varepsilon(x).\end{aligned}$$

Then $\lim_{x\to 0} f(x)$ exists if and only if $a=\frac{9}{4}$ and in this case $\lim_{x\to 0} f(x) = -\frac{1}{12}$.

Exercise 2

1° (a) We have

$$t-4+\frac{12}{2+t}-\frac{8}{(2+t)^2}=\frac{(t-4)(2+t)^2+12(2+t)-8}{(2+t)^2}=\frac{t^3}{(2+t)^2}.$$

(b) $I(x)=\displaystyle\int\frac{\tan^3 x}{(2\cos x+\sin x)^2}\,\mathrm{d}x=\int\frac{\tan^3 x}{(2+\tan x)^2}\times\frac{\mathrm{d}x}{\cos^2 x}$. By setting $t=\tan x$ we obtain $\mathrm{d}t=\dfrac{\mathrm{d}x}{\cos^2 x}$. Then

$$\begin{aligned}I(x)=\int\frac{t^3}{(2+t)^2}\,\mathrm{d}t &= \int\left(t-4+\frac{12}{2+t}-\frac{8}{(2+t)^2}\right)\mathrm{d}t\\ &= \frac{t^2}{2}-4t+12\ln|2+t|+\frac{8}{2+t}+\text{cnst}\end{aligned}$$

where $t=\tan x$.

2° By setting $t=\tan\dfrac{x}{2}$ we obtain $\mathrm{d}x=\dfrac{2\,\mathrm{d}t}{1+t^2}$ and

- $J(x)=\displaystyle\int\frac{\mathrm{d}x}{5+3\cos x}=\int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{5+3\frac{1-t^2}{1+t^2}}=\int\frac{\mathrm{d}t}{4+t^2}=\frac{1}{2}\arctan\frac{t}{2}+\text{cnst}.$
- $K(x)=\displaystyle\int\frac{\mathrm{d}x}{3-5\cos x}=\int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{3-5\frac{1-t^2}{1+t^2}}=-\int\frac{\mathrm{d}t}{1-4t^2}=-\frac{1}{4}\ln\left|\frac{2t+1}{2t-1}\right|+\text{cnst}.$

3° To calculate $L(x)=\displaystyle\int\frac{1}{x}\sqrt{\frac{x}{x+1}}\,\mathrm{d}x$ we can set $t=\sqrt{\dfrac{x}{1+x}}$. We obtain $\dfrac{x}{1+x}=t^2$, then $x=\dfrac{t^2}{1-t^2}$ and $\mathrm{d}x=\dfrac{2t\ \mathrm{d}t}{(1-t^2)^2}$. It follows that

$$L(x)=\int\frac{1-t^2}{t^2}\times t\times\frac{2t}{(1-t^2)^2}\,\mathrm{d}t=2\int\frac{\mathrm{d}t}{1-t^2}=\ln\left|\frac{t+1}{t-1}\right|+\text{cnst}$$

where $t=\sqrt{\dfrac{x}{1+x}}$.

Another method :

$$\begin{aligned}L(x)=2\int\frac{\mathrm{d}x}{\sqrt{4x^2+4x}} &= \int\frac{\mathrm{d}(2x+1)}{\sqrt{(2x+1)^2-1}}\\ &= \ln\left(2x+1+2\sqrt{x^2+x}\right)+\text{cnst}.\end{aligned}$$

Exercise 3

1° The differential equation is written :

$$y' = \frac{\dfrac{y}{x} + \left(\dfrac{y^2}{x^2} - \dfrac{y}{x}\right)\ln\dfrac{y}{x}}{\left(\dfrac{y}{x} - 1\right)\ln\dfrac{y}{x}} \tag{1}$$

It is therefore a homogeneous differential equation. Set $z = \dfrac{y}{x}$, then $y = xz$ and $y' = z + xz'$. Hence (1) is written as :

$$z + xz' = \frac{z + (z^2 - z)\ln z}{(z-1)\ln z} \Leftrightarrow xz' = \frac{z}{(z-1)\ln z} \Leftrightarrow \frac{(z-1)\ln z}{z}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By integration $\displaystyle\int \left(\ln z - \frac{\ln z}{z}\right)\mathrm{d}z = \int \frac{\mathrm{d}x}{x}$.

Then :

$$z\ln z - z - \frac{1}{2}(\ln z)^2 = \ln|x| + \text{cnst} \quad \text{with} \quad z = \frac{y}{x}.$$

2° To calculate $M(x) = \displaystyle\int e^{-\sqrt{x}}\,\mathrm{d}x$ we set $t = \sqrt{x}$, then $x = t^2$ and $\mathrm{d}x = 2t\,\mathrm{d}t$. Therefore

$$M(x) = \int 2te^{-t}\,\mathrm{d}t \underset{\textbf{I.B.P}}{=} -2(t+1)e^{-t} + \text{cnst} = -2(1+\sqrt{x})e^{-\sqrt{x}} + \text{cnst}.$$

Solving the differential equation : $2xy' - \sqrt{x}y = x$. We have

$$y' - \frac{1}{2\sqrt{x}}y = \frac{1}{2} \tag{$*$}$$

It is therefore an LDE. We then look for y in the form $y = uv$. Then $y' = u'v + v'u$. Hence $(*)$ is written as :

$$u'v + \left[v' - \frac{1}{2\sqrt{x}}v\right]u = \frac{1}{2} \tag{$**$}$$

1st step : Calculation of v. We choose v which cancels the bracket :

$$v' - \frac{1}{2\sqrt{x}}v \Leftrightarrow \frac{\mathrm{d}v}{v} = \frac{\mathrm{d}x}{2\sqrt{x}} \Leftrightarrow \int\frac{\mathrm{d}v}{v} = \int\frac{\mathrm{d}x}{2\sqrt{x}} \Leftrightarrow \ln v = \sqrt{x} \Leftrightarrow v = e^{\sqrt{x}}.$$

2nd step : Calculation of u. From $(**)$ we have :

$$u'v = \frac{1}{2} \Leftrightarrow u' = \frac{1}{2}e^{-\sqrt{x}} \Leftrightarrow u = \frac{1}{2}\int e^{-\sqrt{x}}\,\mathrm{d}x = -(1+\sqrt{x})e^{-\sqrt{x}} + C.$$

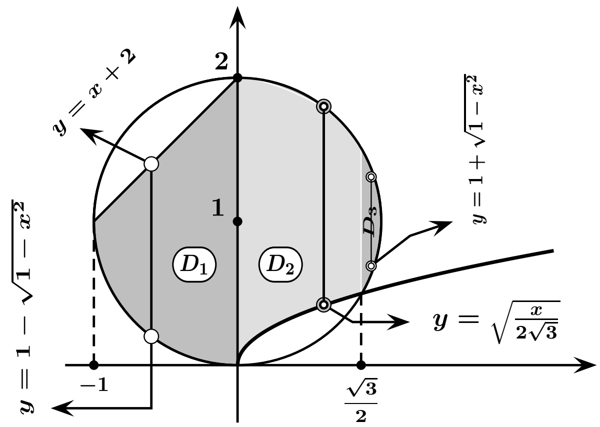
Finally : $y = uv = \left[-(1+\sqrt{x})e^{-\sqrt{x}} + C\right]e^{\sqrt{x}}$.

Exercise 4

1° **Vertically :** $D = D_1 \cup D_2 \cup D_3$:

$$D_1 : \begin{cases} -1 \leq x \leq 0 \\ 1 - \sqrt{1-x^2} \leq y \leq x + 2 \end{cases} \qquad D_2 : \begin{cases} 0 \leq x \leq \dfrac{\sqrt{3}}{2} \\ \sqrt{\dfrac{x}{2\sqrt{3}}} \leq y \leq 1 + \sqrt{1-x^2} \end{cases}$$

$$D_3 : \begin{cases} \dfrac{\sqrt{3}}{2} \leq x \leq 1 \\ 1 - \sqrt{1-x^2} \leq y \leq 1 + \sqrt{1-x^2} \end{cases}$$



Then

$$I = \int_{-1}^{0} \mathrm{d}x \int_{1-\sqrt{1-x^2}}^{2+x} f(x,y)\,\mathrm{d}y + \int_{0}^{\frac{\sqrt{3}}{2}} \mathrm{d}x \int_{\sqrt{\frac{x}{2\sqrt{3}}}}^{1+\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y + \int_{\frac{\sqrt{3}}{2}}^{1} \mathrm{d}x \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y.$$

Horizontally : $D = D_1 \cup D_2 \cup D_3$ with

$$D_1 : \begin{cases} 0 \leq y \leq \dfrac{1}{2} \\ -\sqrt{1-(y-1)^2} \leq x \leq 2\sqrt{3}y^2 \end{cases} \qquad D_2 : \begin{cases} \dfrac{1}{2} \leq y \leq 1 \\ -\sqrt{1-(y-1)^2} \leq x \leq \sqrt{1-(y-1)^2} \end{cases}$$

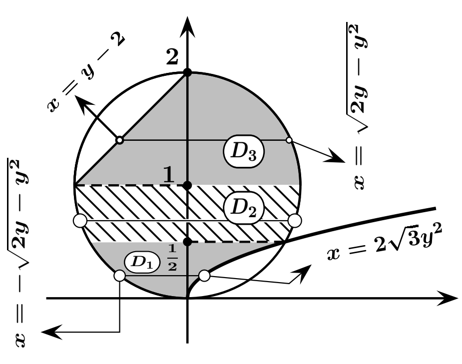
$$D_3 : \begin{cases} 1 \leq y \leq 2 \\ y - 2 \leq x \leq \sqrt{1-(y-1)^2} \end{cases}$$

$I = I_1 + I_2 + I_3$ with :

$$I_1 = \int_{0}^{\frac{1}{2}} \mathrm{d}y \int_{-\sqrt{2y-y^2}}^{2\sqrt{3}y^2} f(x,y)\,\mathrm{d}x;$$

$$I_2 = \int_{\frac{1}{2}}^{1} \mathrm{d}y \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} f(x,y)\,\mathrm{d}x;$$

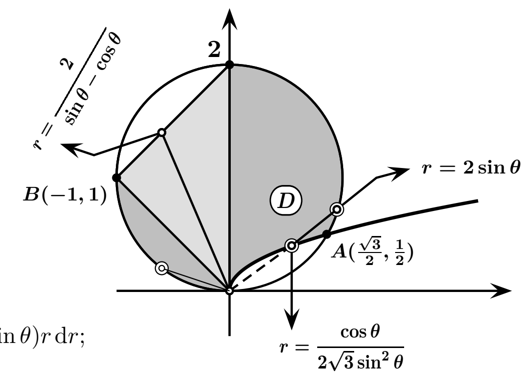
$$I_3 = \int_{1}^{2} \mathrm{d}y \int_{y-2}^{\sqrt{2y-y^2}} f(x,y)\,\mathrm{d}x.$$



By polar coordinates : $D = D_1 \cup D_2 \cup D_3$ with

$$D_1 : \begin{cases} \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2} \\ \dfrac{\cos\theta}{2\sqrt{3}\sin^2\theta} \le r \le 2\sin\theta \end{cases} \qquad D_2 : \begin{cases} \dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{4} \\ 0 \le r \le \dfrac{2}{\sin\theta - \cos\theta} \end{cases}$$

$$D_3 : \begin{cases} \dfrac{3\pi}{4} \le \theta \le \pi \\ 0 \le r \le 2\sin\theta \end{cases}$$



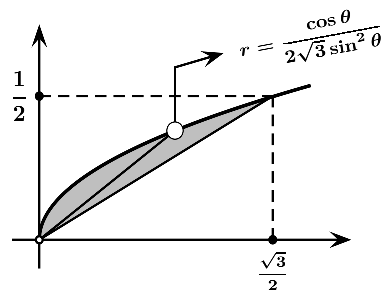
$I = I_1 + I_2 + I_3$ with :

$$I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{\cos\theta}{2\sqrt{3}\sin^2\theta}}^{2\sin\theta} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r;$$

$$I_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \mathrm{d}\theta \int_0^{\frac{2}{\sin\theta-\cos\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r; \quad I_3 = \int_{\frac{3\pi}{4}}^{\pi} \mathrm{d}\theta \int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r.$$

2° The domain Δ is defined in polar coordinates by :

$$\Delta : \begin{cases} \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le \dfrac{\cos\theta}{2\sqrt{3}\sin^2\theta} \end{cases}$$



Then :

$$\begin{aligned} \text{area}(\Delta) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\cos\theta}{2\sqrt{3}\sin^2\theta}} r\,\mathrm{d}r &= \frac{1}{24}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sin^4\theta}\,\mathrm{d}\theta \\ &= \frac{1}{24}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2\theta \times \frac{\mathrm{d}\theta}{\sin^2\theta} \\ &= -\frac{1}{72}\ \cot^3\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{24}. \end{aligned}$$

LEBANESE UNIVERSITY Date: 17 July 2017

Faculty Of Sciences (I)

Final exam **M1109** **Duration: 2 hours**

ANSWER PAGE 159

Exercise 1 **(12 points)**

Let

$$f(x)=\frac{\sin^2(x)\times\ln(1+x)}{(1-e^{2x})(\sqrt{1+x^2}-1)}.$$

1° Give the finite expansion of $f(x)$ to order 2 near $x=0$.

2° Show that f has an extension by continuity at $x=0$. Let g be its extension.

3° Give $g'(0)$.

4° Give the equation of the tangent (T) to the representative (C) of g, and indicate the relative position of (C) with respect to (T) in the neighborhood of 0.

Exercise 2 **(14 points)**

Let

$$f(x)=x\left(\left(2-\cos\frac{1}{x}\right)^{4x^2}-4e^{-2x-\sqrt{4x^2+8x+1}}\right).$$

Determine the equation of the asymptote (T) at $-\infty$ to the curve (C) of f and indicate the relative position of (C) with respect to (T) near $-\infty$.

Exercise 3 **(18 points)**

1° (a) Find a and b so that $1+t^2=(t+2)^2+a(t+2)+b$ for all $t\in\mathbb{R}$.

(b) Calculate $I(t)=\displaystyle\int\frac{1+t^2}{(2+t)^2}\,\mathrm{d}t$. Deduce $J(x)=\displaystyle\int\frac{\mathrm{d}x}{\cos^2x(2\cos x+\sin x)^2}$.

2° By integration by parts, calculate $K(t)=\displaystyle\int\ln\left(\frac{t-1}{t+1}\right)\mathrm{d}t$.

Deduce $L(x)=\displaystyle\int\ln\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\times\frac{\mathrm{d}x}{\sin^2x}$.

3° Calculate $M(x)=\displaystyle\int\frac{\tan(x/2)}{1+3\sin x-\cos x}\,\mathrm{d}x$.

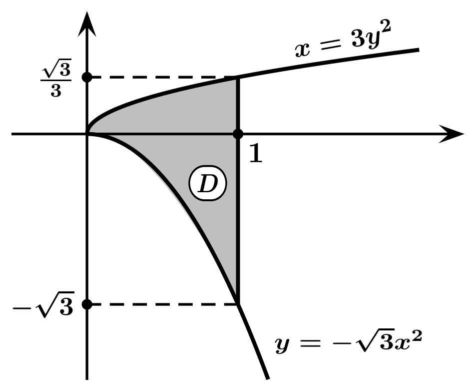
4° Calculate $N(x)=\displaystyle\int\frac{2+\cos^2x}{\sin x}\,\mathrm{d}x$.

Exercise 4 ________________________________ **(6 points)**

Solve the following differential equation : $y' = \dfrac{y\left(1 + 2\ln\dfrac{y}{x}\right)}{x\left(1 + \ln\dfrac{y}{x}\right)}$.

Exercise 5 ________________________________ **(20 points)**

Let D be the shaded domain in the figure below.



1° Let f be a continuous function on D. Write the expression of the double integral $I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ as succession of two simple integrals, by two ways, using cartesian coordinates.

2° Using polar coordinates, calculate the area of the domain D.

Exercise 1

1° There must be simplification by x^3 since the f.e. of $(1-e^{2x})(\sqrt{1+x^2}-1)$ begins by a term of x^3. To find the f.e. of $f(x)$ to order 2 we must then give the f.e. of each usual function to order 5.

- ► $$1-e^{2x} = 1-\left(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}\right)+x^3\varepsilon(x) = -2x-2x^2-\frac{4x^3}{3}+x^3\varepsilon(x).$$

  ► $$\sqrt{1+x^2}-1 = 1+\frac{x^2}{2}-\frac{x^4}{8}-1+x^4\varepsilon(x).$$

  Then

  $$(1-e^{2x})(\sqrt{1+x^2}-1) = \left[-2x-2x^2-\frac{4x^3}{3}\right]\left[\frac{x^2}{2}-\frac{x^4}{8}\right]+x^5\varepsilon(x) = -x^3\left[1+x+\frac{5}{12}x^2\right]+x^5\varepsilon(x).$$

- $$\sin^2 x\ln(1+x) = \left(x-\frac{x^3}{3!}\right)^2\left[x-\frac{x^2}{2}+\frac{x^3}{3}\right]+x^5\varepsilon(x) = \left[x^2-\frac{x^4}{3}\right]\left[x-\frac{x^2}{2}+\frac{x^3}{3}\right]+x^5\varepsilon(x) = x^3-\frac{1}{2}x^4+x^5\varepsilon(x).$$

Then

$$f(x) = -\frac{1-\frac{x}{2}}{1+x+\frac{5x^2}{12}}+x^2\varepsilon(x) = -1+\frac{3}{2}x-\frac{13}{12}x^2+x^2\varepsilon(x).$$

2° $\lim_{x\to 0} f(x) = -1$ then f is e.b.c at $x=0$ by $g(x) = \begin{cases} f(x) & \text{if } x\neq 0 \\ -1 & \text{if } x=0. \end{cases}$

3° $g'(0) = \frac{3}{2}$.

4° The equation of the tangent (T) to the curve (C) of g is $y=-1+\frac{3}{2}x$.
On the other hand, $g(x)-y \underset{0}{\simeq} -\frac{13}{12}x^2 < 0$, then (C) is below (T) in the neighborhood of $x=0$.

Exercise 2

Set $x = \frac{1}{t}$, then $x \to -\infty$ if and only if $t \to 0_-$. Let us determine the f.e. of $tf\left(\frac{1}{t}\right)$ to order 2 near $t = 0_-$. We have

$$tf\left(\frac{1}{t}\right) = e^{\frac{4}{t^2}\ln(2-\cos t)} - 4e^{-\frac{2}{t}-\sqrt{\frac{4}{t^2}+\frac{8}{t}+1}} = e^{\frac{4}{t^2}\ln(2-\cos t)} - 4e^{\frac{-2+\sqrt{4+8t+t^2}}{t}}.$$

In the neighborhood of $t = 0$ and to the order 4 :

$$\begin{aligned}\ln(2-\cos t) &= \ln\left(1+\frac{t^2}{2}-\frac{t^4}{24}\right)+t^4\varepsilon(t) = \frac{t^2}{2}-\frac{t^4}{24}-\frac{1}{2}\left(\frac{t^2}{2}\right)^2+t^4\varepsilon(t)\\ &= \frac{t^2}{2}-\frac{t^4}{6}+t^4\varepsilon(t).\end{aligned}$$

Then

$$e^{\frac{4}{t^2}\ln(2-\cos t)} = e^2 \times e^{-\frac{2t^2}{3}} + t^2\varepsilon(t) = e^2\left[1-\frac{2}{3}t^2\right]+t^2\varepsilon(t).$$

On the other hand, in the neighborhood of $t = 0$ and to the order 3 :

$$\begin{aligned}\sqrt{4+8t+t^2} &= 2\left(1+2t+\frac{t^2}{4}\right)^{\frac{1}{2}}\\ &= 2\left[1+\frac{1}{2}\left(2t+\frac{t^2}{4}\right)-\frac{1}{8}\left(2t+\frac{t^2}{4}\right)^2+\frac{1}{16}(2t)^3\right]+t^3\varepsilon(t)\\ &= 2\left[1+t+\frac{t^2}{8}-\frac{1}{8}\left(4t^2+t^3\right)+\frac{t^3}{2}\right]+t^3\varepsilon(t)\\ &= 2+2t-\frac{3}{4}t^2+\frac{3}{4}t^3+t^3\varepsilon(t).\end{aligned}$$

Then

$$\begin{aligned}e^{\frac{-2+\sqrt{4+8t+t^2}}{t}} = e^{2+\frac{3}{4}(-t+t^2)}+t^2\varepsilon(t) &= e^2\left[1+\frac{3}{4}(-t+t^2)+\frac{1}{2!}\frac{9}{16}(-t)^2\right]+t^2\varepsilon(t)\\ &= e^2\left[1-\frac{3}{4}t+\frac{33}{32}t^2\right]+t^2\varepsilon(t).\end{aligned}$$

It follows that : $tf\left(\frac{1}{t}\right) = e^2\left(-3+3t-\frac{115}{24}t^2\right)+t^2\varepsilon(t)$.

Returning to x, it comes : $f(x) = e^2\left(-3x+3-\frac{115}{24}\times\frac{1}{x}\right)+\frac{1}{x}\varepsilon(x)$.

The equation of the asymptote (T) at $-\infty$ to the (C) is $y = -3e^2(x-1)$.

Since $f(x) - y \underset{-\infty}{\simeq} -\frac{115e^2}{24x} > 0$ then (C) is above (T) near $-\infty$.

Exercise 3

1° (a) We have $1+t^2=(t+2)^2+a(t+2)+b$ for every $t\in\mathbb{R}$ if and only if

$$1+t^2=t^2+(a+4)t+2a+b+4 \qquad \forall t\in\mathbb{R}.$$

By identification we obtain $a+4=0$ and $2a+b+4=1$. Then $a=-4$ and $b=5$.

(b) We have

$$\begin{aligned} I(t) &= \int \frac{(t+2)^2-4(t+2)+5}{(t+2)^2}\,\mathrm{d}t \\ &= \int \mathrm{d}t - 4\int\frac{\mathrm{d}t}{t+1} + 5\int\frac{\mathrm{d}t}{(t+2)^2} \\ &= t-4\ln|2+t|-\frac{5}{2+t}+\text{cnst.} \end{aligned}$$

On the other hand,

$$\begin{aligned} J(x) &= \int \frac{1}{\cos^2 x(2+\tan x)^2}\times\frac{\mathrm{d}x}{\cos^2 x} \\ &= \int \frac{1+\tan^2 x}{(2+\tan x)^2}\times\frac{\mathrm{d}x}{\cos^2 x} \qquad \text{(by setting } t=\tan x\text{)} \\ &= \int \frac{1+t^2}{(t+2)^2}\,\mathrm{d}t = I(t). \end{aligned}$$

Therefore, $J(x)=\tan x-4\ln|2+\tan x|-\dfrac{5}{2+\tan x}+\text{cnst.}$

2° Calculation of $K(t)=\displaystyle\int \ln\left(\frac{t-1}{t+1}\right)\mathrm{d}t$. Integrate by parts with $u=\ln\dfrac{t-1}{t+1}$ and $\mathrm{d}v=\mathrm{d}t$. It comes $v=t$ and $\mathrm{d}u=\dfrac{2}{t^2-1}\,\mathrm{d}t$.

Then

$$K(t)=uv-\int v\,\mathrm{d}u = t\ln\frac{t-1}{t+1}-\int\frac{2t\,\mathrm{d}t}{t^2-1} = t\ln\frac{t-1}{t+1}-\ln(t^2-1)+\text{cnst.}$$

On the other hand, $L(x)=\displaystyle\int \ln\left(\frac{\cot x-1}{\cot x+1}\right)\times\frac{\mathrm{d}x}{\sin^2 x}$. By setting $t=\cot x$, we find $L(x)=-K(t)$.

3° Calculation of $M(x) = \int \frac{\tan(x/2)}{1+3\sin x - \cos x}\,\mathrm{d}x$. By setting $t = \tan\frac{x}{2}$, we obtain $\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$ and

$$M(x) = \int \frac{t}{1+\frac{6t}{1+t^2}-\frac{1-t^2}{1+t^2}}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{3+t} = \ln|3+t| + \text{cnst}.$$

4° $N(x) = \int \frac{2+\cos^2 x}{\sin x}\,\mathrm{d}x = \int \frac{2+\cos^2 x}{\sin^2 x}\sin x\,\mathrm{d}x = \int \frac{2+\cos^2 x}{1-\cos^2 x}\sin x\,\mathrm{d}x.$

By setting $t = \cos x$ it comes $\mathrm{d}t = -\sin x\,\mathrm{d}x$ and

$$N(x) = \int \frac{-2-t^2}{1-t^2}\,\mathrm{d}t = \int\left(1-\frac{3}{1-t^2}\right)\mathrm{d}t = t - \frac{3}{2}\ln\left|\frac{1+t}{1-t}\right| + \text{cnst},$$

with $t = \cos x$.

Exercise 4

we have

$$y' = \frac{\frac{y}{x}\left(1+2\ln\frac{y}{x}\right)}{1+\ln\frac{y}{x}} \tag{1}$$

It is a homogeneous differential equation. By setting $z = \frac{y}{x}$, one obtains $y = xz$ and $y' = z + xz'$. Therefore (1) is written as :

$$z + xz' = \frac{z+2z\ln z}{1+\ln z}.$$

Hence

$$xz' = \frac{z\ln z}{1+\ln z} \Leftrightarrow \frac{1+\ln z}{z\ln z}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By integration :

$$\int \frac{1+\ln z}{z\ln z}\,\mathrm{d}z = \int\frac{\mathrm{d}x}{x} \Leftrightarrow \int\frac{\mathrm{d}(z\ln z)}{z\ln z} = \int\frac{\mathrm{d}x}{x}.$$

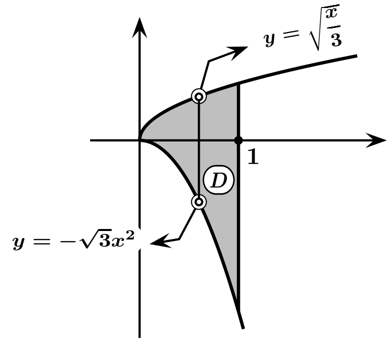
Then

$$\ln|z\ln z| = \ln|x| + \text{cnst} \quad \text{where} \quad z = \frac{y}{x}.$$

Exercise 5

1° **Vertically :**

$$D : \begin{cases} 0 \leq x \leq 1 \\ -\sqrt{3}x^2 \leq y \leq \sqrt{\dfrac{x}{3}} \end{cases}$$



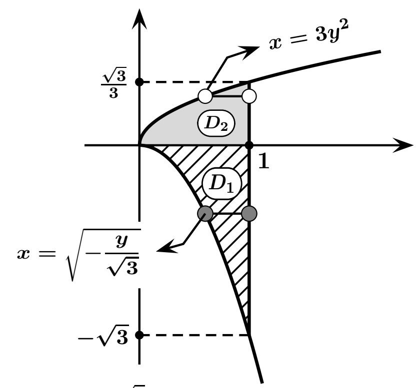
Then :

$$I = \int_0^1 \mathrm{d}x \int_{-\sqrt{3}x^2}^{\sqrt{\frac{x}{3}}} f(x,y)\,\mathrm{d}y.$$

Horizontally : $D = D_1 \cup D_2$:

$$D_1 : \begin{cases} -\sqrt{3} \leq y \leq 0 \\ \sqrt{-\dfrac{y}{\sqrt{3}}} \leq x \leq 1 \end{cases}$$

$$D_2 : \begin{cases} 0 \leq y \leq \dfrac{\sqrt{3}}{3} \\ 3y^2 \leq x \leq 1 \end{cases}$$



Then :

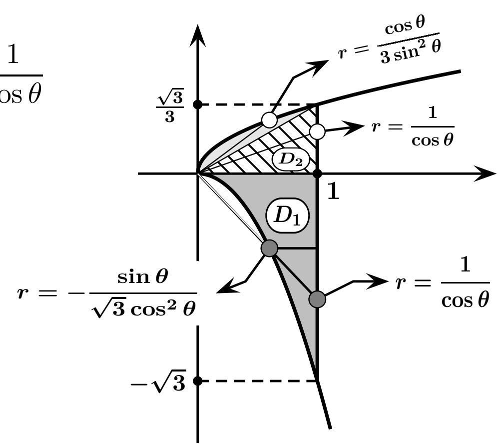
$$I = \int_{-\sqrt{3}}^0 \mathrm{d}y \int_{\sqrt{-\frac{y}{\sqrt{3}}}}^1 f(x,y)\,\mathrm{d}x + \int_0^{\frac{\sqrt{3}}{3}} \mathrm{d}y \int_{3y^2}^1 f(x,y)\,\mathrm{d}x.$$

2° In polar coordinates : $D = D_1 \cup D_2 \cup D_3$ where

$$D_1 : \begin{cases} -\dfrac{\pi}{3} \leq \theta \leq 0 \\ -\dfrac{\sin\theta}{\sqrt{3}\cos^2\theta} \leq r \leq \dfrac{1}{\cos\theta} \end{cases}$$

$$D_2 : \begin{cases} 0 \leq \theta \leq \dfrac{\pi}{6} \\ 0 \leq r \leq \dfrac{1}{\cos\theta} \end{cases}$$

$$D_3 : \begin{cases} \dfrac{\pi}{6} \leq \theta \leq \dfrac{\pi}{2} \\ 0 \leq r \leq \dfrac{\cos\theta}{3\sin^2\theta} \end{cases}$$



By polar coordinates, we have $\mathbf{area(D)} = I_1 + I_2 + I_3$, where

- $$I_1 = \int_{-\frac{\pi}{3}}^{0} \mathrm{d}\theta \int_{-\frac{\sin\theta}{\sqrt{3}\cos^2\theta}}^{\frac{1}{\cos\theta}} r\,\mathrm{d}r \;=\; \frac{1}{2}\int_{-\frac{\pi}{3}}^{0}\left[\frac{1}{\cos^2\theta} - \frac{1}{3}\frac{\tan^2\theta}{\cos^2\theta}\right]\mathrm{d}\theta$$
$$= \frac{1}{2}\left[\tan\theta - \frac{1}{9}\tan^3\theta\right]_{-\frac{\pi}{3}}^{0} = \frac{\sqrt{3}}{3}.$$

- $$I_2 = \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \int_0^{\frac{1}{\cos\theta}} r\,\mathrm{d}r = \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\mathrm{d}\theta}{\cos^2\theta} = \frac{1}{2}\Big[\tan\theta\Big]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{6}.$$

- $$I_3 = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\frac{\cos\theta}{3\sin^2\theta}} r\,\mathrm{d}r = \frac{1}{18}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cot^2\theta}{\sin^2\theta}\mathrm{d}\theta = \frac{1}{54}\Big[-\cot^3\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{18}.$$

Hence : $\mathbf{area(D)} = \dfrac{5}{9}\sqrt{3}.$

Remark : The area of D can be calculated quickly using the cartesian coordinates and by slicing vertically :

$$\begin{aligned}
\mathbf{area(D)} &= \iint_D \mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{-\sqrt{3}x^2}^{\sqrt{\frac{x}{3}}} \mathrm{d}y \\
&= \int_0^1 \left(\frac{\sqrt{x}}{\sqrt{3}} + \sqrt{3}x^2\right)\mathrm{d}x \\
&= \left[\frac{2}{3\sqrt{3}}x\sqrt{x} + \frac{\sqrt{3}}{3}x^3\right]_0^1 = \frac{5\sqrt{3}}{9}.
\end{aligned}$$

LEBANESE UNIVERSITY Date: 22 January 2018
Faculty of Sciences (I)

Final Exam **M1109** Duration: 2 hours

ANSWER PAGE 167

Exercise 1 **(10 points)**

Let

$$f(x) = \frac{\ln(1 + (e^{x^2} - 1)\sin x)}{x(\sqrt{\cos x} - 1)}.$$

1° Write down the finite expansion of f of order 2 near 0.

2° Prove that f is extendable by continuity at $x = 0$. Let g be its extension.

3° Give $g'(0)$ and the equation of the tangent line to the curve of g at $x = 0$, then study their relative position.

Exercise 2 **(6 points)**

Let

$$f(x) = x\left(e^{\frac{1}{x}} - \sqrt[3]{\operatorname{ch}\frac{1}{x}} + \ln\frac{x+1}{x}\right).$$

Find the equation of the asymptote to the curve of f and study their relative near $+\infty$.

Exercise 3 **(18 points)**

1° Calculate $I(t) = \displaystyle\int \frac{\mathrm{d}t}{t^2(4 + 2t^2)}$. Deduce $J(x) = \displaystyle\int \frac{\mathrm{d}x}{\tan^2\left(\frac{x}{2}\right)(3 + \cos x)}$.

2° Calculate $K(x) = \displaystyle\int \frac{1}{1-x}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x$.

3° Calculate $L(x) = \displaystyle\int \frac{\sin x \cos^2 x}{1 + \sin^2 x}\,\mathrm{d}x$.

4° Justify the equality $1 + \cos x \sin x = \dfrac{1}{2}[1 + (\sin x + \cos x)^2]$ and use it to calculate the value of $M = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{1 + \cos x \sin x}\,\mathrm{d}x$.

Exercise 4 **(12 points)**

Solve the following differential equations

1° $y\sqrt{x}\,y' = \dfrac{y^2}{\sqrt{x}} - \dfrac{y^2\sqrt{y}}{x}$.

2° $\tan\left(\dfrac{y}{2}\right)(\cos x)y' = \sin y \sin^2 x$.

Exercise 5 **(6 points)**

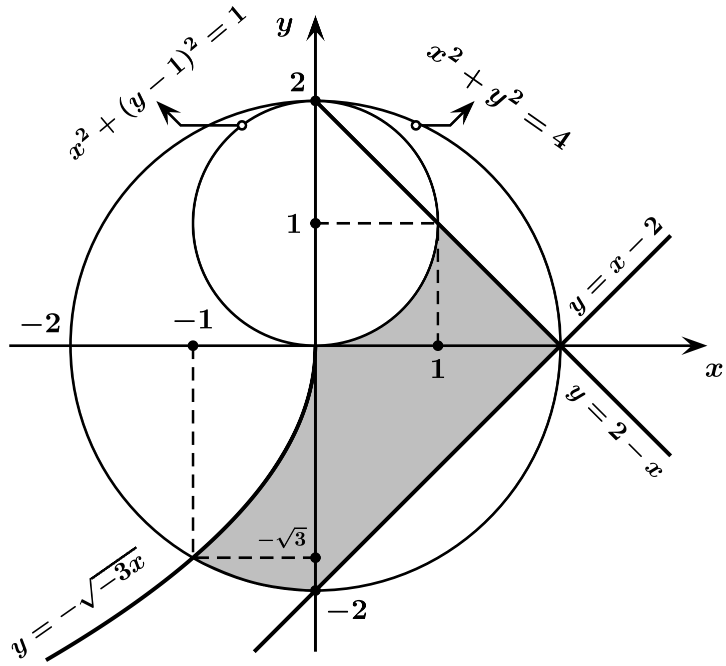
Calculate the following double integral : $\displaystyle\int_0^1 \mathrm{d}y \int_0^y \cos(x^2 - 2x)\,\mathrm{d}x$.

Exercise 6 **(18 points)**

Consider the double integral

$$I = \iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

where D is the shaded domain in the figure below.



1° Using the **cartesian coordinates**, write down the double integral I by two different ways, as succession of two simple integrals.

2° Using the **polar coordinates**, rewrite the double integral I as succession of two simple integrals.

Exercise 1

1° There must be simplification by x^3 since the f.e. of $x(\sqrt{\cos x}-1)$ begins with a term in x^3. To find the f.e. of $f(x)$ to order 2 at $x=0$ we must give the f.e. of each usual function to order 5.

- $$\begin{aligned}\sqrt{\cos x} &= \left[1+\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)\right]^{\frac{1}{2}}+x^4\varepsilon(x)\\ &= 1+\frac{1}{2}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)-\frac{1}{8}\left(-\frac{x^2}{2}+\frac{x^4}{24}\right)^2+x^4\varepsilon(x)\\ &= 1-\frac{x^2}{4}-\frac{x^4}{96}+x^4\varepsilon(x).\end{aligned}$$

  Then
  $$x(\sqrt{\cos x}-1)=-\frac{x^3}{4}\left(1+\frac{x^2}{24}\right)+x^5\varepsilon(x).$$

- We have
  $$\begin{aligned}\underbrace{(e^{x^2}-1)}_{\text{order }4}\times\underbrace{\sin x}_{\text{order }3} &= \left(1+x^2+\frac{x^4}{2}-1\right)\left(x-\frac{x^3}{6}\right)+x^5\varepsilon(x)\\ &= x^3+\frac{x^5}{3}+x^5\varepsilon(x).\end{aligned}$$

  Then
  $$\ln(1+(e^{x^2}-1)\sin x)=\ln\left(1+\left(x^3+\frac{x^5}{3}\right)\right)+x^5\varepsilon(x)=x^3+\frac{x^5}{3}+x^5\varepsilon(x).$$

Hence : $f(x)=\dfrac{x^3+\dfrac{x^5}{3}}{-\dfrac{1}{4}\left(x^3+\dfrac{x^5}{24}\right)}+x^2\varepsilon(x)=-4-\dfrac{7x^2}{6}+x^2\varepsilon(x).$

2° $\lim\limits_{x\to 0}f(x)=-4$, then f is extendable by continuity at $x=0$ by the function g defined by :

$$g(0)=-4,\qquad\text{and}\qquad g(x)=f(x)\quad\text{if}\quad x\neq 0.$$

3° We have $g'(0)=0$. On the other hand, the equation of the tangent (T) to the (C) curve of g is $y=-4$. Since $f(x)-y\underset{0}{\simeq}-\dfrac{7x^2}{6}<0$ then (C) is below (T).

Exercise 2

Let's put $x = \dfrac{1}{t}$ and give the f.e. of $tf\left(\dfrac{1}{t}\right)$ to the order 2 in $t = 0$. We have :

$$\begin{aligned} tf\left(\frac{1}{t}\right) &= e^t - \sqrt[3]{\mathrm{ch(t)}} + \ln(1+t) \\ &= 1 + t + \frac{t^2}{2} - \left(1 + \frac{t^2}{2}\right)^{\frac{1}{3}} + t - \frac{t^2}{2} + t^2\varepsilon(t) \\ &= 1 + t + \frac{t^2}{2} - 1 - \frac{t^2}{6} + t - \frac{t^2}{2} + t^2\varepsilon(t) \\ &= 2t - \frac{t^2}{6} + t^2\varepsilon(t). \end{aligned}$$

Returning to x, we get :

$$f(x) = 2 - \frac{1}{6x} + \frac{1}{x}\varepsilon(x).$$

The equation of the asymptote (D) is then $y = 2$. Since $f(x) - y \underset{+\infty}{\simeq} -\dfrac{1}{6x} < 0$ then the curve of f is below (D).

Exercise 3

1° We have

$$\frac{1}{t^2(2+t^2)} = \frac{1}{2} \times \frac{2}{t^2(2+t^2)} = \frac{1}{2} \times \frac{2+t^2-t^2}{t^2(2+t^2)} = \frac{1}{2}\left(\frac{1}{t^2} - \frac{1}{2+t^2}\right).$$

Then

$$I(t) = \frac{1}{2}\int \frac{\mathrm{d}t}{t^2(2+t^2)} = \frac{1}{4}\int\left(\frac{1}{t^2} - \frac{1}{2+t^2}\right)\mathrm{d}t = -\frac{1}{4t} - \frac{1}{4\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + \text{cnst}.$$

Let's calculate now $J(x) = \displaystyle\int \frac{\mathrm{d}x}{\tan^2\left(\frac{x}{2}\right)(3+\cos x)}$.

Let $t = \tan\left(\dfrac{x}{2}\right)$, then $\mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$ and since $\cos x = \dfrac{1-t^2}{1+t^2}$ we get :

$$J(x) = \int \frac{\frac{2\,\mathrm{d}t}{1+t^2}}{t^2\left(3 + \frac{1-t^2}{1+t^2}\right)} = 2\int \frac{\mathrm{d}t}{t^2(4+2t^2)} = 2I(t).$$

2° Calculation of $K(x)=\displaystyle\int \frac{1}{1-x}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x$.

1st method. We can put $t=\sqrt{\dfrac{x}{1-x}}$. So $\dfrac{x}{1-x}=t^2$ and $x=\dfrac{t^2}{1+t^2}$, then $\mathrm{d}x=\dfrac{2t\ \mathrm{d}t}{(1+t^2)^2}$. Hence

$$K(x)=\int (1+t^2)\times t\times \frac{2t}{(1+t^2)^2}\,\mathrm{d}t = 2\int \frac{t^2\,\mathrm{d}t}{1+t^2} = 2(t-\arctan t)+\text{cnst}.$$

2nd method. Let $x=\sin^2\theta$ with $\theta\in\left]0,\frac{\pi}{2}\right[$, then $\mathrm{d}x=2\sin\theta\cos\theta\,\mathrm{d}\theta$ and

$$\begin{aligned} K(x) &= \int \frac{1}{1-\sin^2\theta}\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}2\sin\theta\cos\theta\,\mathrm{d}\theta = 2\int \tan^2\theta\,\mathrm{d}\theta \\ &= 2\int\left(\frac{1}{\cos^2\theta}-1\right)\mathrm{d}\theta = 2(\tan(\theta)-\theta)+\text{cnst}. \end{aligned}$$

3rd method. We write $K(x)=\displaystyle\int \sqrt{x}\times\frac{\mathrm{d}x}{(1-x)^{\frac{3}{2}}}$. Integrate by parts with $u=\sqrt{x}$ and $\mathrm{d}v=\dfrac{\mathrm{d}x}{(1-x)^{\frac{3}{2}}}$. We get then $\mathrm{d}u=\dfrac{\mathrm{d}x}{2\sqrt{x}}$, $v=\dfrac{2}{\sqrt{1-x}}$ and

$$\begin{aligned} K(x)=2\sqrt{\frac{x}{1-x}}-\int\frac{\mathrm{d}x}{\sqrt{x-x^2}} &= 2\sqrt{\frac{x}{1-x}}-\int\frac{\mathrm{d}x}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}} \\ &= 2\sqrt{\frac{x}{1-x}}-\arcsin(2x-1)+\text{cnst}. \end{aligned}$$

3° Calculation of $L(x)=\displaystyle\int\frac{\sin x\cos^2 x}{1+\sin^2 x}\,\mathrm{d}x$. We have

$$\begin{aligned} L(x)=\int\frac{\cos^2 x}{2-\cos^2 x}\sin x\,\mathrm{d}x \underset{t=\cos x}{=} \int\frac{-t^2}{2-t^2}\,\mathrm{d}t &= \int\left(1-\frac{2}{2-t^2}\right)\mathrm{d}t \\ &= t-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+\text{cnst}. \end{aligned}$$

4° $1+(\sin x+\cos x)^2 = 1+\sin^2 x+\cos^2 x+2\cos x\sin x = 2(1+\sin x\cos x)$. Then

$$\begin{aligned} M=\int_0^{\frac{\pi}{4}}\frac{2(\cos x-\sin x)\,\mathrm{d}x}{1+(\cos x+\sin x)^2} &\underset{t=\cos x+\sin x}{=} \int_1^{\sqrt{2}}\frac{2\,\mathrm{d}t}{1+t^2} \\ &= 2\arctan t\Big]_1^{\sqrt{2}} = 2\arctan(\sqrt{2})-\frac{\pi}{2}. \end{aligned}$$

Exercise 4

1° The differential equation $y\sqrt{x}y' = \dfrac{y^2}{\sqrt{x}} - \dfrac{y^2\sqrt{y}}{x}$ is written

$$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^{3/2} \tag{1}$$

It is therefore a homogeneous differential equation. Let's put $z = \dfrac{y}{x}$, then $y = xz$ and $y' = z + xz'$. So (1) is written :

$$z + xz' = z - z^{3/2} \Leftrightarrow xz' = -z^{3/2} \Leftrightarrow -z^{-3/2}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By integration, we find $\dfrac{2}{\sqrt{z}} = \ln|x| + \text{cnst}$ where $z = \dfrac{y}{x}$.

2° The differential equation $\tan\left(\dfrac{y}{2}\right)\cos x\, y' = \sin y \sin^2 x$ is of separable variables which is written :

$$\frac{\tan\left(\frac{y}{2}\right)}{\sin y}\,\mathrm{d}y = \frac{\sin^2 x}{\cos x}\,\mathrm{d}x.$$

But $\displaystyle\int \frac{\tan\left(\frac{y}{2}\right)}{\sin y}\,\mathrm{d}y = \int \frac{\mathrm{d}y}{2\cos^2\left(\frac{y}{2}\right)} = \tan\left(\frac{y}{2}\right) + \text{cnst},$

and $\displaystyle\int \frac{\sin^2 x}{\cos x}\,\mathrm{d}x = \int \frac{\sin^2 x}{1-\sin^2 x}\cos x\,\mathrm{d}x \underset{t=\sin x}{=} \int \frac{t^2}{1-t^2}\,\mathrm{d}t = -t + \frac{1}{2}\ln\frac{1+t}{1-t} + \text{cnst}.$

Finally, we have : $\tan\left(\dfrac{y}{2}\right) = -\sin x + \dfrac{1}{2}\ln\dfrac{1+\sin x}{1-\sin x} + \text{cnst}.$

Exercise 5

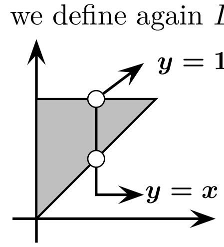
To calculate the integral double

$$I = \int_0^1 \mathrm{d}y \int_0^y \cos(x^2 - 2x)\,\mathrm{d}x$$

one must change the order of integration. The D integration domain is defined by :

$$D : \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$$

By changing the order of integration we define again D :



$$\boldsymbol{D} : \begin{cases} \boldsymbol{0 \le x \le 1} \\ \boldsymbol{x \le y \le 1} \end{cases}$$

Then : $I = \displaystyle\int_0^1 \mathrm{d}x \int_x^1 \cos(x^2 - 2x)\,\mathrm{d}y = \int_0^1 \cos(x^2-2x)(1-x)\,\mathrm{d}x$. By putting $t = x^2 - 2x$ we obtain :

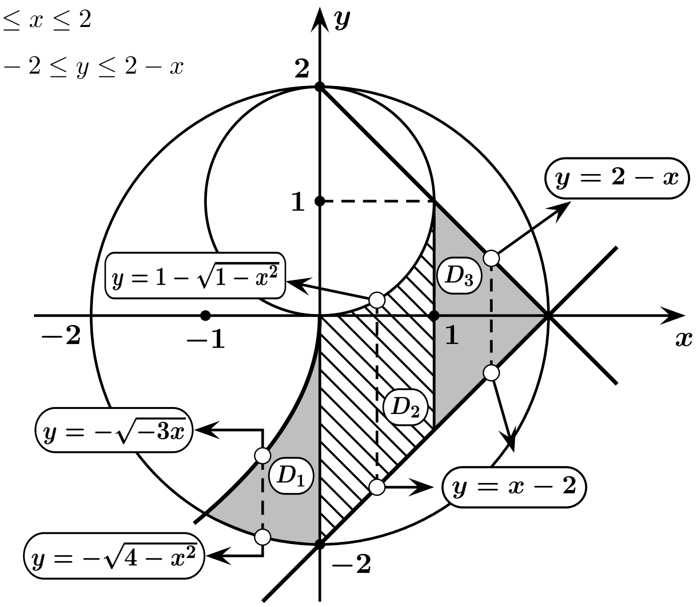
$$I = -\frac{1}{2}\int_0^{-1} \cos(t)\,\mathrm{d}t = \frac{1}{2}\sin(1).$$

Exercise 6

1° **Vertically :** $D = D_1 \cup D_2 \cup D_3$:

$$D_1 : \begin{cases} -1 \leq x \leq 0 \\ -\sqrt{4-x^2} \leq y \leq -\sqrt{-3x} \end{cases} \qquad D_2 : \begin{cases} 0 \leq x \leq 1 \\ x-2 \leq y \leq 1-\sqrt{1-x^2} \end{cases}$$

$$D_3 : \begin{cases} 1 \leq x \leq 2 \\ x-2 \leq y \leq 2-x \end{cases}$$



$$I = \int_{-1}^{0} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{-\sqrt{-3x}} f(x,y)\,\mathrm{d}y + \int_{0}^{1} \mathrm{d}x \int_{x-2}^{1-\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y + \int_{1}^{2} \mathrm{d}x \int_{x-2}^{2-x} f(x,y)\,\mathrm{d}y.$$

Horizontally : $D = D_1 \cup D_2 \cup D_3$ with

$$D_1 : \begin{cases} -2 \leq y \leq -\sqrt{3} \\ -\sqrt{4-y^2} \leq x \leq 2+y \end{cases} \qquad D_2 : \begin{cases} -\sqrt{3} \leq y \leq 0 \\ -\dfrac{y^2}{3} \leq x \leq 2+y \end{cases}$$

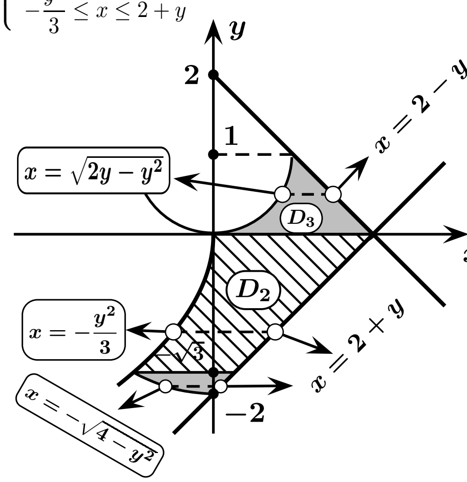
$$D_3 : \begin{cases} 0 \leq y \leq 1 \\ \sqrt{2y-y^2} \leq x \leq 2-y \end{cases}$$

$I = I_1 + I_2 + I_3$ with :

$$I_1 = \int_{-2}^{-\sqrt{3}} \mathrm{d}y \int_{-\sqrt{4-y^2}}^{2+y} f(x,y)\,\mathrm{d}x;$$

$$I_2 = \int_{-\sqrt{3}}^{0} \mathrm{d}y \int_{-\frac{y^2}{3}}^{2+y} f(x,y)\,\mathrm{d}x;$$

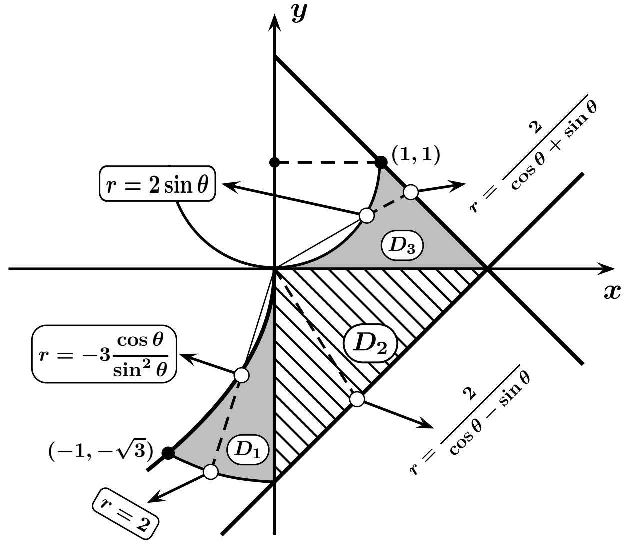
$$I_3 = \int_{0}^{1} \mathrm{d}y \int_{\sqrt{2y-y^2}}^{2-y} f(x,y)\,\mathrm{d}x.$$



2° **By polar coordinates :** $D = D_1 \cup D_2 \cup D_3$ with

$$D_1 : \begin{cases} \dfrac{4\pi}{3} \le \theta \le \dfrac{3\pi}{2} \\ \dfrac{-3\cos\theta}{\sin^2\theta} \le r \le 2 \end{cases} \qquad D_2 : \begin{cases} \dfrac{3\pi}{2} \le \theta \le 2\pi \\ 0 \le r \le \dfrac{2}{\cos\theta - \sin\theta} \end{cases}$$

$$D_3 : \begin{cases} 0 \le \theta \le \dfrac{\pi}{4} \\ 2\sin\theta \le r \le \dfrac{2}{\cos\theta + \sin\theta} \end{cases}$$



$I = I_1 + I_2 + I_3$ with :

$$I_1 = \int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} \mathrm{d}\theta \int_{\frac{-3\cos\theta}{\sin^2\theta}}^{2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r;$$

$$I_2 = \int_{\frac{3\pi}{2}}^{2\pi} \mathrm{d}\theta \int_0^{\frac{2}{\cos\theta - \sin\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r;$$

$$I_3 = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_{2\sin\theta}^{\frac{2}{\cos\theta + \sin\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r.$$

LEBANESE UNIVERSITY Date: 24 January 2019

Faculty Of Sciences (I)

Final Exam **M1109** **Time: 2 hours**

SOLUTION PAGE 175

Exercise 1 [12 points]

Let
$$f(x)=\sin(2x)+e^{x-\frac{5x^2}{8}}+\sqrt{1+\ln\left(1+x+x^2\right)}.$$

1. Give the $FE_3(0)$ of $f(x)$.
2. Give the equation of the tangent (T) to the curve (C) of the function f at the point of abscissa $x=0$ and indicate the position of (T) relative to (C) in the neighborhood of $x=0$.

Exercise 2 [8 points]

Let
$$f(x)=x\left[\ln\left(2-\text{ch}\frac{1}{x}\right)-\sqrt{e}\times e^{x+\sqrt{1+x+x^2}}\right].$$

Give the equation of the asymptote (D) to the representative curve (C) of f in the neighborhood of $-\infty$ and indicate the position of (D) relative to (C) in the neighborhood of $-\infty$.

Exercise 3 [15 points]

1. (a) Calculate the following integral :
$$I(z)=\int\frac{\mathrm{d}z}{\sqrt{z^2+2z+5}}.$$

   (b) Solve the following differential equation :
$$xy'=y+\sqrt{y^2+5x^2+2xy},\quad \text{with } x>0.$$

2. Solve the following differential equation :
$$e^xy'-\frac{e^x}{1+e^{-x}}\,y=1.$$

Exercise 4 ——— [15 points]

1. (a) Calculate $I(t) = \int \frac{dt}{t^2(2+t)}$.

   (b) Calculate $J(x) = \int \frac{dx}{\tan^2 x(2\cos x + \sin x)\cos x}$.

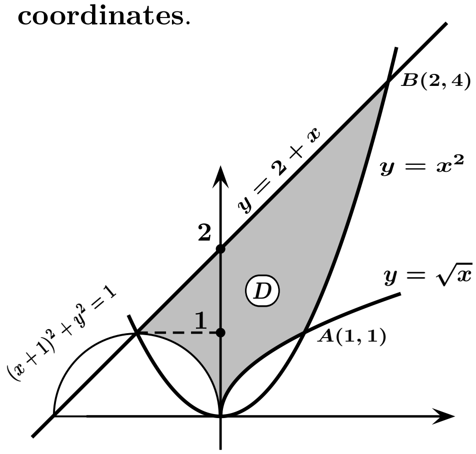
2. Calculate $M = \int x(\ln x)^2\, dx$.

3. (a) Calculate $K(t) = \int \frac{(t+1)\,dt}{1+2t-t^2}$.

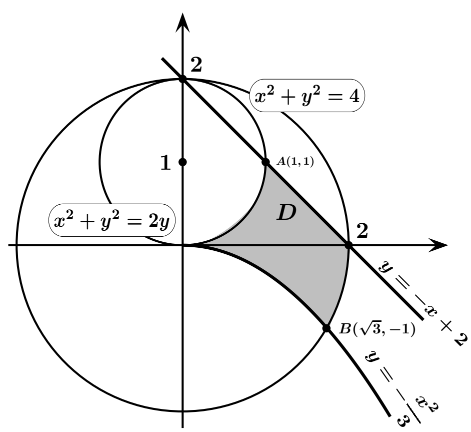
   (b) Calculate $L(x) = \int \frac{\tan\left(\frac{x}{2}\right)+1}{(\tan x + 1)\cos x}\, dx$.

Exercise 5 ——— [20 points]

1. Let D be the shaded area shown below.
Write $I = \iint_D f(x,y)\, dx\, dy$ as a succession of two simple integrals in two different ways, using the **cartesian coordinates**.



2. Let D be the shaded area shown below.
Write $I = \iint_D f(x,y)\, dx\, dy$ as a succession of two simple integrals, using **polar coordinates**.



Exercise 1

1. In the neighborhood of $x = 0$ and to the order 3 :
   - $\sin(2x) = 2x - \dfrac{(2x)^3}{3!} + x^3\varepsilon(x) = 2x - \dfrac{4}{3}x^3 + x^3\varepsilon(x)$.
   - $$e^{x-\frac{5x^2}{8}} = 1 + x - \frac{5x^2}{8} + \frac{1}{2}\left(x - \frac{5x^2}{8}\right)^2 + \frac{1}{6}x^3 + x^3\varepsilon(x)$$
     $$= 1 + x - \frac{1}{8}x^2 - \frac{11}{24}x^3 + x^3\varepsilon(x).$$
   - $$\ln(1 + x + x^2) = x + x^2 - \frac{1}{2}(x + x^2)^2 + \frac{1}{3}x^2 + x^3\varepsilon(x)$$
     $$= x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + x^3\varepsilon(x).$$
   - $$\sqrt{1+\ln(1+x+x^2)} = 1 + \frac{1}{2}\left(x + \frac{1}{2}x^2 - \frac{2}{3}x^3\right) - \frac{1}{8}\left(x + \frac{1}{2}x^2\right)^2 + \frac{1}{16}x^3 + x^3\varepsilon(x)$$
     $$= 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{19}{48}x^3 + x^3\varepsilon(x)$$

   So : $f(x) = 2 + \dfrac{7}{2}x - \dfrac{35}{16}x^3 + x^3\varepsilon(x)$.
2. The equation of (T) is : $y = 2 + \dfrac{7}{2}x$. Since, $f(x) - y \underset{0}{\simeq} -\dfrac{35}{16}x^3$, then :
   - If $x > 0$ then $f(x) - y < 0$ and (C) is below (T).
   - If $x < 0$ then $f(x) - y > 0$ and (C) is above (T).

Exercise 2

Let $x = 1/t$. Then $x \to -\infty \Leftrightarrow t \to 0_-$. Let's give the f.e. of

$$tf\left(\frac{1}{t}\right) = \ln(2 - \text{ch}t) - \sqrt{e} \times e^{\frac{1-\sqrt{1+t+t^2}}{t}}$$

to the order 2 near $t = 0_-$. We have :

- $\ln(2 - \text{ch}t) = \ln\left(1 - \dfrac{t^2}{2}\right) = -\dfrac{t^2}{2} + t^2\varepsilon(t)$.
- $$\sqrt{1+t+t^2} = 1 + \frac{1}{2}(t + t^2) - \frac{1}{8}(t + t^2)^2 + \frac{1}{16}t^3 + t^3\varepsilon(t)$$
  $$= 1 + \frac{1}{2}t + \frac{3}{8}t^2 - \frac{3}{16}t^3 + t^3\varepsilon(t).$$
- $e^{\frac{1-\sqrt{1+t+t^2}}{t}} = e^{-\frac{1}{2} - \frac{3}{8}t + \frac{3}{16}t^2} + t^2\varepsilon(t) = e^{-\frac{1}{2}}\left(1 - \dfrac{3}{8}t + \dfrac{33}{128}t^2\right) + t^2\varepsilon(t)$.

Hence :

$$tf\left(\frac{1}{t}\right) = -1 + \frac{3}{8}t - \frac{97}{128}t^2 + t^2\varepsilon(t).$$

Let's go back to x :

$$f(x) = -x + \frac{3}{8} - \frac{97}{128x} + \frac{1}{x}\varepsilon(x).$$

The equation of (D) is then $y = -x+\frac{3}{8}$ and since $f(x) - y \underset{-\infty}{\simeq} -\frac{97}{128x} > 0$ then (C) is above (D) in the neighborhood of $-\infty$.

Exercise 3

1. (a) $I = \int \frac{\mathrm{d}z}{\sqrt{4+(z+1)^2}} = \ln\left(z+1+\sqrt{4+(z+1)^2}\right) + \text{cnst}.$

   (b) We write :

   $$y' = \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 2\frac{y}{x} + 5} \qquad (*)$$

   It is a homogeneous differential equation. Let $z = \frac{y}{x}$, then $y = xz$ and $y' = z + xz'$. So $(*)$ becomes :

   $$z + xz' = z + \sqrt{z^2 + 2z + 5}.$$

   Then

   $$\frac{\mathrm{d}z}{\sqrt{z^2+2z+5}} = \frac{\mathrm{d}x}{x}.$$

   By integration :

   $$\ln(z+1+\sqrt{4+(z+1)^2}) = \ln|x| + \text{cnst} \quad \text{with} \quad z = \frac{y}{x}.$$

2. **Resolution of the differential equation :** $e^x y' - \frac{e^x}{1+e^{-x}}\, y = 1$.

   We have

   $$y' - \frac{1}{1+e^{-x}}y = \frac{1}{e^x} \qquad (1)$$

   It is a linear differential equation of order 1. We are looking for y in the form $y = uv$. Then $y' = u'v + v'u$. Thus (1) is written as :

   $$u'v + \left[v' - \frac{1}{1+e^{-x}}v\right]u = \frac{1}{e^x} \qquad (2)$$

First step : Calculation of v. We choose v which cancels the bracket :

$$v' - \frac{1}{1+e^{-x}}v = 0 \Leftrightarrow \frac{dv}{v} = \frac{e^x\,dx}{1+e^x} \quad \Rightarrow \quad \ln v = \ln(1+e^x)$$

$$\Rightarrow \quad v = 1 + e^x.$$

Second step : Calculation of u. From (2) we get :

$$u'v = \frac{1}{e^x} \Leftrightarrow u' = \frac{1}{e^x(1+e^x)} \Leftrightarrow u = \int \frac{1}{e^x(1+e^x)}\,dx.$$

Let's calculate u. We can write :

$$\begin{aligned} u = \int \frac{1}{e^x(1+e^x)}\,dx &= \int \frac{1+e^x-e^x}{e^x(1+e^x)}\,dx \\ &= \int \frac{1}{e^x}\,dx - \int \frac{1}{1+e^x}\,dx \\ &= \int e^{-x}\,dx - \int \frac{-e^{-x}}{1+e^{-x}}\,dx \\ &= -e^{-x} + \ln(1+e^{-x}) + \text{cnst}. \end{aligned}$$

Finally : $y = uv = (1+e^x)(-e^{-x} - x + \ln(1+e^x) + \text{cnst})$.

Exercise 4

1. (a) We have

$$\begin{aligned} I(t) &= \frac{1}{2}\int \frac{2}{t^2(2+t)}\,dt = \frac{1}{2}\int \frac{2+t-t}{t^2(2+t)}\,dt \\ &= \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{2}\int \frac{1}{t(2+t)}\,dt \\ &= \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{4}\int \frac{2+t-t}{t(2+t)}\,dt \\ &= \frac{1}{2}\int \frac{dt}{t^2} - \frac{1}{4}\int \frac{dt}{t} + \frac{1}{4}\int \frac{dt}{2+t}. \end{aligned}$$

   Hence : $I(t) = -\frac{1}{2t} - \frac{1}{4}\ln|t| + \frac{1}{4}\ln|2+t| + \text{cnst}$.

   (b) $J(x) = \int \frac{dx}{\tan^2 x(2\cos x + \sin x)\cos x} = \int \frac{dx}{\tan^2 x(2+\tan x)\cos^2 x}$.

   Then, by setting $t = \tan x$, we obtain $dt = \frac{dx}{\cos^2 x}$ and then $J(x) = I(t)$.

2. **Calculation of $M = \int x(\ln x)^2\,dx$.** Integrate by parts with

$$\begin{cases} u = (\ln x)^2 \\ dv = x\,dx \end{cases} \Rightarrow \begin{cases} du = \frac{2}{x}(\ln x)\,dx \\ v = \frac{x^2}{2} \end{cases}$$

then

$$\begin{aligned} M = uv - \int v\,du &= \frac{x^2}{2}(\ln x)^2 - \int x\ln x\,dx \\ &= \frac{x^2}{2}(\ln x)^2 - K, \end{aligned}$$

where

$$K = \int x\ln x\,dx \underset{I.P.P}{=} \frac{x^2}{2}\ln x - \frac{x^2}{4} + \text{cnst}.$$

3. (a) **Calculation of $K(t) = \int \frac{(t+1)\,dt}{1+2t-t^2}$.** We have :

$$\begin{aligned} K(t) &= -\frac{1}{2}\int \frac{-2t+2-4}{1+2t-t^2}\,dt \\ &= -\frac{1}{2}\int \frac{-2t+2}{1+2t-t^2}\,dt + 2\int \frac{d(t-1)}{(\sqrt{2})^2-(t-1)^2} \\ &= -\frac{1}{2}\ln|1+2t-t^2| + \frac{1}{\sqrt{2}}\ln\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right| + \text{cnst}. \end{aligned}$$

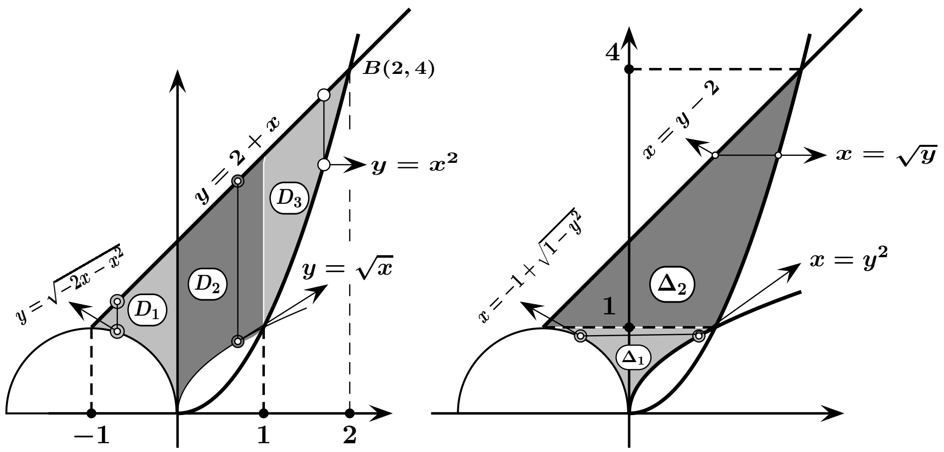
(b) **Calculation of $L(x) = \int \frac{\tan\left(\frac{x}{2}\right)+1}{(\tan x+1)\cos x}\,dx$.**

By setting $t = \tan\frac{x}{2}$, we get $dx = \frac{2}{1+t^2}\,dt$. Then

$$L(x) = \int \frac{t+1}{\left(\frac{2t}{1-t^2}+1\right)\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt = 2K(t).$$

Exercise 5

1.



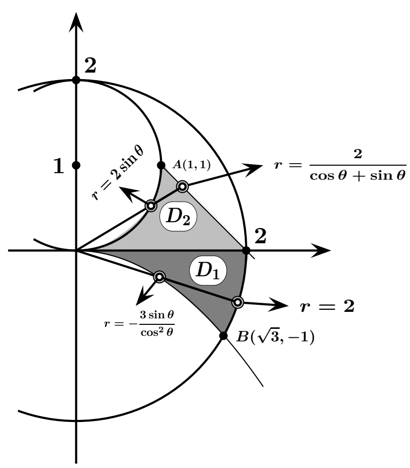
Vertically :

$$I = \int_{-1}^{0} \mathrm{d}x \int_{\sqrt{-2x-x^2}}^{2+x} f(x,y)\,\mathrm{d}y + \int_{0}^{1} \mathrm{d}x \int_{\sqrt{x}}^{2+x} f(x,y)\,\mathrm{d}y + \int_{1}^{2} \mathrm{d}x \int_{x^2}^{2+x} f(x,y)\,\mathrm{d}y.$$

Horizontally :

$$I = \int_{0}^{1} \mathrm{d}y \int_{-1+\sqrt{1-y^2}}^{y^2} f(x,y)\,\mathrm{d}x + \int_{1}^{4} \mathrm{d}y \int_{y-2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x.$$

2.



In Polar coordinates

$I = I_1 + I_2$ where

$$I_1 = \int_{-\frac{\pi}{6}}^{0} \mathrm{d}\theta \int_{-\frac{3\sin\theta}{\cos^2\theta}}^{2} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r$$

$$I_2 = \int_{0}^{\frac{\pi}{4}} \mathrm{d}\theta \int_{2\sin\theta}^{\frac{2}{\cos\theta+\sin\theta}} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r$$

LEBANESE UNIVERSITY Date: 05 September 2019
Faculty of Sciences (I)

Final Exam **M1109** Time: 2 hours

SOLUTION PAGE 182

Exercise 1 [12 points]
Let $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$. Consider the function f defined by :

$$f(x) = \frac{\sqrt[3]{6+\sqrt{3+\cos(4x)}} - e^{-ax^2} - \operatorname{ch}(\sqrt{b}\,x) + \frac{x^2}{6}}{x^4}.$$

Determine a and b so that $\lim_{x\to 0} f(x) = -\frac{1}{24}$.

Exercise 2 [8 points]
Let $f(x) = e^{\frac{1-\cos(\sqrt{2}\,x)}{x^2}} \left(\frac{\ln(e+x)-1}{x}\right)$.

1. Give the finite expansion of $f(x)$ at the order 2 in the neighborhood of 0.
2. Show that f is extendable by continuity at $x = 0$. Let g be its extension.
3. Give the equation of the tangent (D) at the curve (C) of g at the point of abscissa $x = 0$ and indicate the relative position of (D) and (C) in the neighborhood of 0.

Exercise 3 [15 points]

1. Calculate $I(x) = \displaystyle\int \frac{dx}{(3+\cos^2 x)\tan x}$.
2. (a) Calculate $J(x) = \displaystyle\int \ln^2(x)\frac{dx}{x^2}$.
   (b) Deduce $K(t) = \displaystyle\int t^2 e^{-t}\,dt$.
3. (a) Factorize $s^2 + 3s + 2$.
   (b) Deduce $L(s) = \displaystyle\int \frac{ds}{\sqrt{(4+2s)(1+s)}}$ and $M(t) = \displaystyle\int \frac{t\,dt}{\sqrt{(4+2t^2)(1+t^2)}}$.
   (c) Calculate $N(x) = \displaystyle\int \frac{\tan\frac{x}{2}}{\sqrt{3+\cos x}}\,dx$.

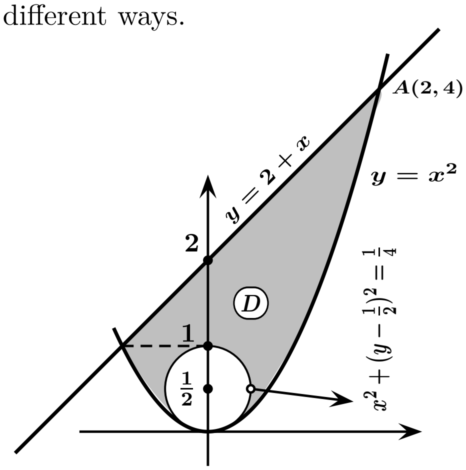
Exercise 4 ______________________________ [15 points]

Solve the following differential equations :

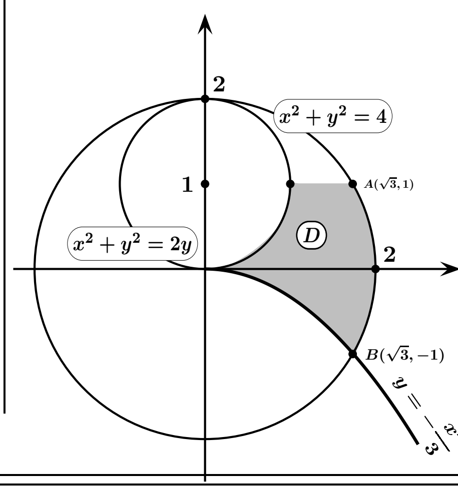
1. $\boldsymbol{y' + \frac{1}{x}y = \frac{1}{\sqrt{1+x}}}$.
2. $\boldsymbol{y' = \dfrac{y^2 e^{\frac{y}{x}} + x^2\sqrt{e^{\frac{y}{x}}+1}}{yxe^{\frac{y}{x}}}}$.

Exercise 5 ______________________________ [20 points]

1. Let $\boldsymbol{D}$ be the shaded area shown below.
Using cartesian coordinates, write down $\boldsymbol{I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y}$ as succession of two simple integrals by two different ways.



2. Let $\boldsymbol{D}$ be the shaded area shown below.
Using polar coordinates, write down $\boldsymbol{I = \iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y}$ as succession of two simple integrals.



LU Sciences (I) Year 2018-2019	SOLUTION	Calculus M1109

Exercise 1

In the neighborhood of $x = 0$ and at to order 4 :

- $\cos(4x) = 1 - 8x^2 + \dfrac{32}{3}x^4 + x^4\varepsilon(x)$.
- $$\begin{aligned}\sqrt{3+\cos(4x)} &= \sqrt{4 - 8x^2 + \frac{32}{3}x^4} + x^4\varepsilon(x)\\ &= 2\sqrt{1 - 2x^2 + \frac{8}{3}x^4} + x^4\varepsilon(x)\\ &= 2\left(1 + \frac{1}{2}\left(-2x^2 + \frac{8}{3}x^4\right) - \frac{1}{8}(-2x^2)^2\right) + x^4\varepsilon(x)\\ &= 2 - 2x^2 + \frac{5}{3}x^4 + x^4\varepsilon(x).\end{aligned}$$
- $$\begin{aligned}\sqrt[3]{6+\sqrt{3+\cos(4x)}} &= \sqrt[3]{8 - 2x^2 + \frac{5}{3}x^4} + x^4\varepsilon(x)\\ &= 2\sqrt[3]{1 - \frac{1}{4}x^2 + \frac{5}{24}x^4} + x^4\varepsilon(x)\\ &= 2\left(1 + \frac{1}{3}\left(-\frac{1}{4}x^2 + \frac{5}{24}x^4\right) - \frac{1}{9}\left(-\frac{1}{4}x^2\right)^2\right) + x^4\varepsilon(x)\\ &= 2 - \frac{1}{6}x^2 + \frac{1}{8}x^4 + x^4\varepsilon(x).\end{aligned}$$
- $e^{-ax^2} = 1 - ax^2 + \dfrac{a^2}{2}x^4 + x^4\varepsilon(x)$.
- $\mathrm{ch}(\sqrt{b}x) = 1 + \dfrac{b}{2}x^2 + \dfrac{b^2}{24}x^4 + x^4\varepsilon(x)$.

Hence :

$$f(x) = \frac{a - \frac{b}{2}}{x^2} - \frac{a^2}{2} - \frac{b^2}{24} + \frac{1}{8} + \varepsilon(x).$$

It follows that $\lim\limits_{x\to 0} f(x) = -\dfrac{1}{24}$ if and only if

$$\begin{cases} a - \dfrac{b}{2} = 0 \\ -\dfrac{a^2}{2} - \dfrac{b^2}{24} + \dfrac{1}{8} = -\dfrac{1}{24} \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{b}{2} \\ -\dfrac{b^2}{6} = -\dfrac{1}{6} \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{b}{2} \\ b^2 = 1 \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{b}{2} \\ b = \pm 1. \end{cases}$$

Then : $b = 1$ and $a = \dfrac{1}{2}$ since b is positive.

Exercise 2

1. In the neighborhood of $x = 0$:
   - $\cos(\sqrt{2}\,x) = 1 - x^2 + \dfrac{1}{6}x^4 + x^4\varepsilon(x)$.
   - $e^{\frac{1-\cos(\sqrt{2}x)}{x^2}} = e^{1-\frac{1}{6}x^2} + x^2\varepsilon(x) = e \times \left(1 - \dfrac{1}{6}x^2\right) + x^2\varepsilon(x).$
   - $$\begin{aligned}\ln(e+x) = &\ \ln\left(e\left(1+\frac{x}{e}\right)\right) = \ln e + \ln\left(1+\frac{x}{e}\right)\\ = &\ 1 + \frac{x}{e} - \frac{1}{2}\frac{x^2}{e^2} + \frac{1}{3}\frac{x^3}{e^3} + x^3\varepsilon(x).\end{aligned}$$
   - $\dfrac{\ln(e+x)-1}{x} = \dfrac{1}{e} - \dfrac{1}{2}\dfrac{x}{e^2} + \dfrac{1}{3}\dfrac{x^2}{e^3} + x^2\varepsilon(x).$

   Then
   $$f(x) = 1 - \frac{x}{2e} + \left(\frac{1}{3e^2} - \frac{1}{6}\right)x^2 + x^2\varepsilon(x).$$
2. $\lim\limits_{x\to 0} f(x) = 1$, then f is e.b.c at $x = 0$ with $g(x) = \begin{cases} f(x) & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$
3. $y = 1 - \dfrac{x}{2e}$ and since $g(x) - y \underset{0}{\simeq} \left(\dfrac{1}{3e^2} - \dfrac{1}{6}\right)x^2 < 0$, then (C) is below (D) in the neighborhood of $x = 0$.

Exercise 3

1. **Calculation of the integral $I(x) = \displaystyle\int \frac{1}{(3+\cos^2 x)\tan x}$.**

   First method : We write
   $$I(x) = \int \frac{\cos x\,\mathrm{d}x}{(3+\cos^2 x)\sin x} = \int \frac{\cos x\,\mathrm{d}x}{(4-\sin^2 x)\sin x}.$$
   Set $t = \sin x$ then $\mathrm{d}t = \cos x\ \mathrm{d}x$ and
   $$\begin{aligned} I(x) = \int \frac{1}{(4-t^2)t}\,\mathrm{d}t &= \frac{1}{4}\int \frac{4-t^2+t^2}{(4-t^2)t}\,\mathrm{d}t \\ &= \frac{1}{4}\int \frac{1}{t}\,\mathrm{d}t + \frac{1}{4}\int \frac{t}{4-t^2}\,\mathrm{d}t \\ &= \frac{1}{4}\ln|t| - \frac{1}{8}\ln|4-t^2| + \text{cnst}, \end{aligned}$$
   where $t = \sin x$.

Second method : We write

$$I(x) = \int \frac{dx}{(\frac{3}{\cos^2 x} + 1)\tan x} \times \frac{dx}{\cos^2 x} = \int \frac{d(\tan x)}{(3(1+\tan^2 x)+1)\tan x}.$$

By setting $t = \tan x$, we obtain

$$\begin{aligned} I(x) = \int \frac{1}{(4+3t^2)t}\,dt &= \frac{1}{4}\int \frac{4+3t^2-3t^2}{(4+3t^2)t}\,dt \\ &= \frac{1}{4}\int \frac{1}{t}\,dt - \frac{3}{4}\int \frac{t}{4+3t^2}\,dt \\ &= \frac{1}{4}\ln|t| - \frac{1}{8}\ln(4+3t^2) + \text{cnst}, \end{aligned}$$

where $t = \tan x$.

Third method : We write

$$I(x) = \int \frac{\sin x \cos x}{(3+\cos^2 x)\sin^2 x}\,dx = \int \frac{\sin x \cos x}{(4-\sin^2 x)\sin^2 x}\,dx.$$

Set $t = \sin^2 x$, then $dt = 2\sin x \cos x\, dx$ and

$$\begin{aligned} I(t) &= \frac{1}{2}\int \frac{1}{(4-t)t}\,dt \\ &= \frac{1}{8}\int \frac{1}{t}\,dt + \frac{1}{8}\int \frac{1}{4-t}\,dt \\ &= \frac{1}{8}\ln|t| - \frac{1}{8}\ln|4-t| + \text{cnst}, \end{aligned}$$

where $t = \sin^2 t$.

2. (a) **Calculation of** $J(x) = \int \ln^2(x)\frac{dx}{x^2}$ **:** Integrating by parts with $u = \ln^2 x$ and $dv = \frac{dx}{x^2}$, then $du = 2\ln(x)\frac{dx}{x}$ and $v = -\frac{1}{x}$. We obtain then

$$J(x) = uv - \int v\,du = -\frac{1}{x}\ln^2(x) + 2\int \ln(x)\frac{dx}{x^2}.$$

Also, by integrating by parts, we find

$$\int \ln(x)\frac{dx}{x^2} = -\frac{1}{x}\ln(x) + \int \frac{dx}{x^2} = -\frac{1}{x}\ln(x) - \frac{1}{x} + \text{cnst}.$$

(b) **Calculation of the integral** $K(t) = \int t^2 e^{-t}\,dt$ **:** Set $t = \ln x$, then $dt = \dfrac{dx}{x}$ and $e^{-t} = \dfrac{1}{x}$, hence

$$K(t) = \int \ln^2(x) \times \frac{1}{x} \times \frac{dx}{x} = J(x).$$

3. (a) $s^2 + 3s + 2 = (s+1)(s+2)$.

(b) **Calculation of** $L(s) = \int \dfrac{ds}{\sqrt{(4+2s)(1+s)}}$ **:** We have :

$$\begin{aligned} L(s) &= \frac{1}{\sqrt{2}} \int \frac{ds}{\sqrt{(s+2)(s+1)}} \\ &= \frac{1}{\sqrt{2}} \int \frac{ds}{\sqrt{s^2+3s+2}} \\ &= \frac{1}{\sqrt{2}} \int \frac{ds}{\sqrt{\left(s+\frac{3}{2}\right)^2 - \frac{1}{4}}} \\ &= \frac{1}{\sqrt{2}} \ln\left(s + \frac{3}{2} + \sqrt{s^2+3s+2}\right) + \text{cnst}. \end{aligned}$$

Calculation of $M(t) = \int \dfrac{t\,dt}{\sqrt{(4+2t^2)(1+t^2)}}$: Set $s = t^2$, then $ds = 2t\,dt$ and

$$M(t) = \frac{1}{2} \int \frac{ds}{\sqrt{(4+2s)(1+s)}} = \frac{1}{2} L(s).$$

(c) **Calculation of** $N(x) = \int \dfrac{\tan\frac{x}{2}}{\sqrt{3+\cos x}}\,dx$ **:** Set $t = \tan\dfrac{x}{2}$, then $dx = \dfrac{2}{1+t^2}\,dt$. We obtain then

$$\begin{aligned} N(x) &= \int \frac{t}{\sqrt{3 + \frac{1-t^2}{1+t^2}}} \times \frac{2\,dt}{1+t^2} \\ &= \int \frac{2t\,dt}{(1+t^2)\sqrt{\frac{4+2t^2}{1+t^2}}} = 2\int \frac{t\,dt}{\sqrt{(4+2t^2)(1+t^2)}} = 2M(t). \end{aligned}$$

Exercise 4

1. **Solving the differential equation :**

$$\boldsymbol{y' + \frac{1}{x}y = \frac{1}{\sqrt{1+x}}} \tag{1}$$

It is a LDE. We then look for $\boldsymbol{y}$ in the form $\boldsymbol{y = uv}$.
Then $\boldsymbol{y' = u'v + v'u}$. So (1) is written as :

$$\boldsymbol{u'v + \left[v' + \frac{1}{x}v\right] u = \frac{1}{\sqrt{1+x}}} \tag{2}$$

First step : Calculation of $\boldsymbol{v}$. We choose $\boldsymbol{v}$ which cancels the brackets :

$$\boldsymbol{v' + \frac{1}{x}v = 0 \Leftrightarrow \frac{\mathrm{d}v}{v} = -\frac{\mathrm{d}x}{x} \Rightarrow \ln v = -\ln x \Rightarrow v = \frac{1}{x}.}$$

Second step : Calculation of $\boldsymbol{u}$. From (2) we have :

$$\boldsymbol{u'v = \frac{1}{\sqrt{1+x}} \Leftrightarrow u = \int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x.}$$

Calculation of $\boldsymbol{u = \int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x}$.

First method : Set $\boldsymbol{t = \sqrt{1+x}}$ then $\boldsymbol{x = t^2 - 1}$ and $\boldsymbol{\mathrm{d}x = 2t\,\mathrm{d}t}$. Hence

$$\boldsymbol{u = 2\int (t^2-1)\,\mathrm{d}t = 2\left(\frac{t^3}{3} - t\right) + k = 2\left(\frac{1}{3}(1+x)^{\frac{3}{2}} - \sqrt{1+x}\right) + k,}$$

where $\boldsymbol{k} \in \mathbb{R}$.

Second method : $\boldsymbol{\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x = \int \frac{(x+1)-1}{\sqrt{1+x}}\,\mathrm{d}x}$

$$\boldsymbol{= \int \left(\sqrt{1+x} - \frac{1}{\sqrt{1+x}}\right)\mathrm{d}x \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + k.}$$

Third method : Integrating by parts with $\boldsymbol{u = x}$ and $\boldsymbol{\mathrm{d}v = \frac{\mathrm{d}x}{\sqrt{1+x}}}$. Then $\boldsymbol{\mathrm{d}u = \mathrm{d}x}$ and $\boldsymbol{v = 2\sqrt{1+x}}$ and

$$\boldsymbol{\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x = 2x\sqrt{1+x} - 2\int \sqrt{1+x}\,\mathrm{d}x = 2x\sqrt{1+x} - \frac{4}{3}(1+x)^{\frac{3}{2}} + k.}$$

Finally : $\boldsymbol{y = uv = 2\frac{1}{x}\left(\frac{1}{3}(1+x)^{\frac{3}{2}} - \sqrt{1+x} + k\right)}$.

2. **Solving** $\boldsymbol{y' = \dfrac{y^2 e^{\frac{y}{x}} + x^2\sqrt{e^{\frac{y}{x}}+1}}{yxe^{\frac{y}{x}}}}$.

We write

$$\boldsymbol{y' = \frac{\frac{y^2}{x^2}e^{\frac{y}{x}} + \sqrt{e^{\frac{y}{x}}+1}}{\frac{y}{x}e^{\frac{y}{x}}}} \tag{1}$$

It is a homogeneous differential equation. Set $\boldsymbol{z = \dfrac{y}{x}}$, then $\boldsymbol{y = xz}$ and $\boldsymbol{y' = z + xz'}$. So (1) is written as :

$$\boldsymbol{z + xz' = \frac{z^2e^z + \sqrt{e^z+1}}{ze^z} \Leftrightarrow xz' = \frac{\sqrt{1+e^z}}{ze^z}.}$$

We obtain then,

$$\boldsymbol{\frac{ze^z\,\mathrm{d}z}{\sqrt{1+e^z}} = \frac{\mathrm{d}x}{x}.}$$

By integration :

$$\boldsymbol{\int \frac{ze^z\,\mathrm{d}z}{\sqrt{1+e^z}} = \ln|x| + \text{cnst}.}$$

Calculation of $\boldsymbol{\displaystyle\int \frac{ze^z\,\mathrm{d}z}{\sqrt{1+e^z}}}$ **:** Integrating by parts with $\boldsymbol{u = z}$ and $\boldsymbol{\mathrm{d}v = \dfrac{e^z\,\mathrm{d}z}{\sqrt{1+e^z}}}$. We find :

$$\boldsymbol{\int \frac{ze^z\,\mathrm{d}z}{\sqrt{1+e^z}} = 2z\sqrt{1+e^z} - 2K,}$$

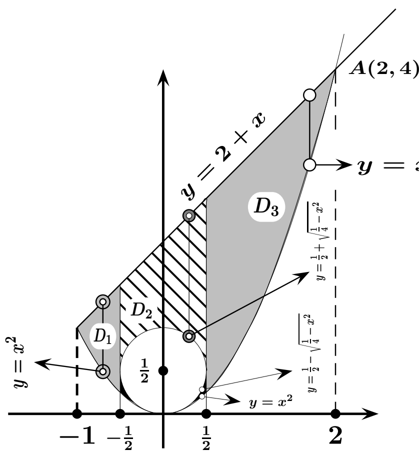
where

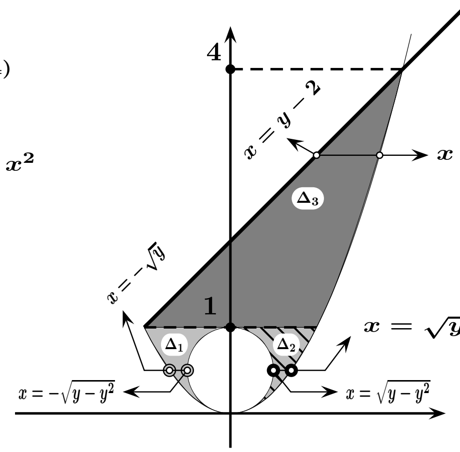
$$\begin{aligned}\boldsymbol{K = \int\sqrt{1+e^z}\,\mathrm{d}z} \quad &\underset{t=e^z}{=} \quad \boldsymbol{\int \frac{\sqrt{1+t}}{t}\,\mathrm{d}t}\\ &\underset{s=\sqrt{1+t}}{=} \quad \boldsymbol{2\int \frac{s^2}{s^2-1}\,\mathrm{d}s}\\ &= \quad \boldsymbol{2\int\left(1 - \frac{1}{1-s^2}\right)\mathrm{d}s}\\ &= \quad \boldsymbol{2\left(s - \frac{1}{2}\ln\left|\frac{1+s}{1-s}\right|\right) + \text{cnst},}\end{aligned}$$

where $\boldsymbol{s = \sqrt{1+e^z}}$.

Exercise 5

1.





Vertically : $D = D_1 \cup D_2 \cup D_3 \cup D_4$ and

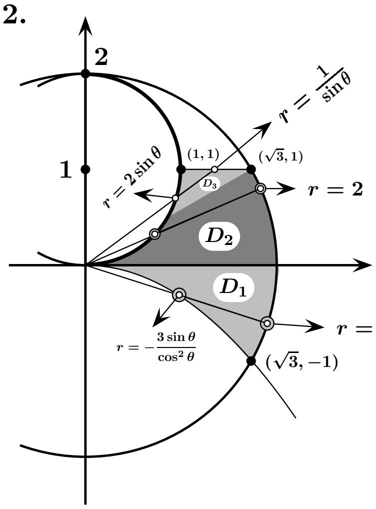
$$I = \int_{-1}^{-\frac{1}{2}} dx \int_{x^2}^{2+x} f(x,y)\,dy + \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2}}^{2+x} f(x,y)\,dy$$

$$+\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{x^2}^{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}} f(x,y)\,dy + \int_{\frac{1}{2}}^{2} dx \int_{x^2}^{2+x} f(x,y)\,dy.$$

Horizontally : $D = \Delta_1 \cup \Delta_2 \cup \Delta_3$ and

$$I = \int_0^1 dy \int_{-\sqrt{y}}^{-\sqrt{y-y^2}} f(x,y)\,dx + \int_0^1 dy \int_{\sqrt{y-y^2}}^{\sqrt{y}} f(x,y)\,dx + \int_1^4 dy \int_{y-2}^{\sqrt{y}} f(x,y)\,dx.$$

2.



In polar coordinates :

$I = I_1 + I_2 + I_3$ where

$$I_1 = \int_{-\frac{\pi}{6}}^{0} d\theta \int_{-\frac{3\sin\theta}{\cos^2\theta}}^{2} f(r\cos\theta, r\sin\theta)r\,dr$$

$$I_2 = \int_{0}^{\frac{\pi}{6}} d\theta \int_{2\sin\theta}^{2} f(r\cos\theta, r\sin\theta)r\,dr$$

$$I_3 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta \int_{2\sin\theta}^{\frac{1}{\sin\theta}} f(r\cos\theta, r\sin\theta)r\,dr$$

LEBANESE UNIVERSITY Date: 21 February 2020
Faculty Of Sciences (I)

Final Exam **M1109** **Time: 2 hours**

SOLUTION PAGE 191

Exercise 1 [10 points]

Let

$$f(x) = \frac{x \ln\left(\frac{1+\sin x}{1-\operatorname{sh} x}\right)}{e^{x^2}-1}.$$

1. Give the finite expansion of $f(x)$ to order 2 in the neighborhood of $x = 0$.
2. Show that f is extendable by continuity to $x = 0$. Let g be its extension.
3. Give $g'(0)$.
4. Determine the equation of the tangent (D) to the curve (C) of the function g at the point of abscissa $x = 0$ and precise the relative position of (D) with respect to (C) in the neighborhood of $x = 0$.

Exercise 2 [12 points]

Let

$$f(x) = x^2\left[\ln\left(x^2\left(e^{\frac{1}{2x^2}} - \cos\left(\frac{1}{x}\right)\right)\right) + \frac{\sqrt{1-8x+4x^2}}{x} + 2\right].$$

Determine the equation of the asymptote (D) to the curve (C) of f at $-\infty$ and precise the relative position of (D) with respect to (C) at $-\infty$.

Exercise 3 [15 points]

1. (a) Calculate $I(x) = \displaystyle\int \frac{e^x}{2e^x+1}\,\mathrm{d}x$ and $J(x) = \displaystyle\int \frac{1}{\sqrt{1+2e^x}}\,\mathrm{d}x$.

   (b) Solve the differential equation : $e^{-x}y' - \dfrac{1}{1+2e^x}y = e^{-x}$.

2. Solve the following differential equation :

$$x\left(\tan\frac{y}{x}\right)^2 y' = x\left(\left(\cos\frac{y}{x}\right)^2 + 1\right) + y\left(\tan\frac{y}{x}\right)^2.$$

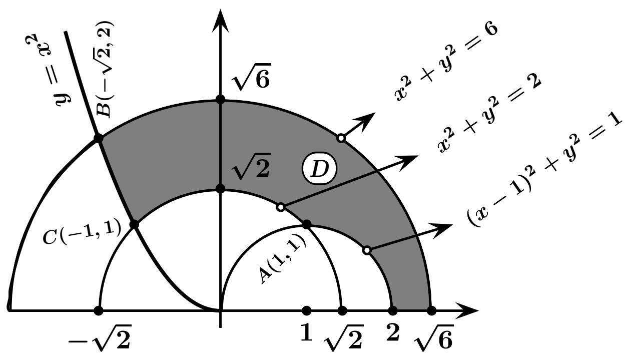
Exercise 4 ______________________________ [10 points]

1. Calculate $\boldsymbol{I(t)} = \displaystyle\int \frac{\mathbf{d}\boldsymbol{t}}{\mathbf{1} + \boldsymbol{t} - \boldsymbol{t}^{\mathbf{2}}}$.
2. Calculate $\boldsymbol{J(x)} = \displaystyle\int \boldsymbol{x}\frac{\cos \boldsymbol{x} - 2\sin \boldsymbol{x}}{(2\cos \boldsymbol{x} + \sin \boldsymbol{x})^2}\,\mathbf{d}\boldsymbol{x}$.

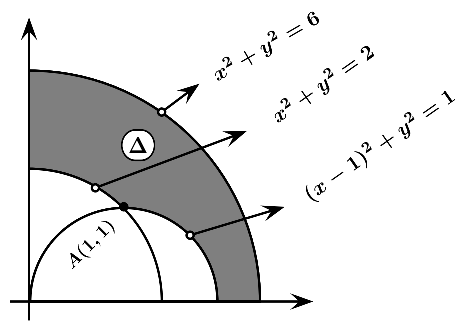
Exercise 5 ______________________________ [23 points]

1° Let $\boldsymbol{D}$ be the shaded domain in the figure below.

Write down $\boldsymbol{I} = \displaystyle\iint_{\boldsymbol{D}} \boldsymbol{f(x,y)}\,\mathbf{d}\boldsymbol{x}\,\mathbf{d}\boldsymbol{y}$ as a succession of two simple integrals in two different ways, by using **cartesian**.



2° Let $\boldsymbol{\Delta}$ be the shaded domain in the figure below.



Write down $\boldsymbol{I} = \displaystyle\iint_{\boldsymbol{\Delta}} \boldsymbol{f(x,y)}\,\mathbf{d}\boldsymbol{x}\,\mathbf{d}\boldsymbol{y}$ as a succession of two simple integrals, by using **polar coordinates**.

SOLUTION

Exercise 1

1. To find the f.e. of $f(x) = \dfrac{x\ln\left(\frac{1+\sin x}{1-\mathbf{sh}\,x}\right)}{e^{x^2}-1}$ to l'order 2 in the neighborhood of $x=0$, we give the f.e. of each usual function to order 4 since we have a simplification by x^2. In the neighborhood of $x=0$,

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + x^4\varepsilon(x).$$

- $$\begin{aligned}\ln(1+\sin x) &= \ln\left(1 + x - \frac{x^3}{6}\right) + x^3\varepsilon(x)\\ &= x - \frac{x^3}{6} - \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{3}x^3 + x^3\varepsilon(x)\\ &= x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + x^3\varepsilon(x).\end{aligned}$$
- $$\begin{aligned}\ln(1-\mathbf{sh}\,x) &= \ln\left(1 + (-x - \frac{x^3}{6})\right) + x^3\varepsilon(x)\\ &= -x - \frac{x^3}{6} - \frac{1}{2}\left(x + \frac{x^3}{6}\right)^2 + \frac{1}{3}(-x)^3 + x^3\varepsilon(x)\\ &= -x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + x^3\varepsilon(x).\end{aligned}$$

Then $\ln\left(\dfrac{1+\sin x}{1-\mathbf{sh}\,x}\right) = 2x + \dfrac{2}{3}x^3 + x^3\varepsilon(x)$. So

$$f(x) = \frac{2 + \frac{2}{3}x^2}{1 + \frac{1}{2}x^2} + x^2\varepsilon(x) = 2 - \frac{1}{3}x^2 + x^2\varepsilon(x).$$

2. Since $\lim\limits_{x\to 0} f(x) = 2$, then f is e.b.c at $x=0$ by the function g defined by $g(x) = \begin{cases} f(x) & \text{if} \quad x \neq 0;\\ 2 & \text{if} \quad x = 0.\end{cases}$
3. $g'(0) = 0$.
4. The equation of the tangent (D) at the curve (C) of the function g at the point of abscissa $x=0$ is $y=2$. Since, $f(x) - y \underset{0}{\simeq} -\dfrac{1}{3}x^2 \leq 0$, then (C) is below (D) in the neighborhood of $x=0$.

Exercise 2

Let $x = \dfrac{1}{t}$ then $x \to -\infty \Leftrightarrow t \to 0_-$ and

$$tf\left(\frac{1}{t}\right) = \frac{1}{t}\left[\ln \frac{e^{\frac{1}{2}t^2} - \cos t}{t^2} - \sqrt{4 - 8t + t^2} + 2\right].$$

Let's us giving the f.e. of $tf\left(\dfrac{1}{t}\right)$ to order 2 in the neighborhood of $t = 0_-$.
We have :

- $e^{\frac{1}{2}t^2} = 1 + \dfrac{1}{2}t^2 + \dfrac{1}{8}t^4 + t^5\varepsilon(t)$.
- $\cos t = 1 - \dfrac{1}{2}t^2 + \dfrac{1}{24}t^4 + t^5\varepsilon(t)$.

Then

$$\ln \frac{e^{\frac{1}{2}t^2} - \cos t}{t^2} = \ln\left(1 + \frac{1}{12}t^2\right) + t^3\varepsilon(t) = \frac{1}{12}t^2 + t^3\varepsilon(t).$$

In the ohter hand, we have :

$$\begin{aligned}
\sqrt{4 - 8t + t^2} &= 2\left[1 + \left(-2t + \frac{1}{4}t^2\right)\right]^{\frac{1}{2}} \\
&= 2\left[1 + \frac{1}{2}\left(-2t + \frac{1}{4}t^2\right) - \frac{1}{8}\left(-2t + \frac{1}{4}t^2\right)^2 + \frac{1}{16}(-2t)^3\right] + t^3\varepsilon(t) \\
&= 2 - 2t - \frac{3}{4}t^2 - \frac{3}{4}t^3 + t^3\varepsilon(t).
\end{aligned}$$

Hence

$$tf\left(\frac{1}{t}\right) = 2 + \frac{5}{6}t + \frac{3}{4}t^2 + t^2\varepsilon(t).$$

Returning to x, we obtain :

$$f(x) = 2x + \frac{5}{6} + \frac{3}{4x} + \frac{1}{x}\varepsilon(x).$$

The equation of (D) is then $y = 2x + \dfrac{5}{6}$, and since $f(x) - y \underset{-\infty}{\simeq} \dfrac{3}{4x} < 0$ then (C) is below (D) in the neighborhood of $-\infty$.

Exercise 3

1. (a) **Calculation of $I(x) = \displaystyle\int \frac{e^x}{2e^x+1}\,\mathrm{d}x$.** We have

$$I(x) = \frac{1}{2}\int \frac{2e^x}{2e^x+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(2e^x+1)}{2e^x+1} = \frac{1}{2}\ln(2e^x+1) + \text{cnst}.$$

Calculation of $J(x) = \displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+2e^x}}$. Let $t = e^x$ then $x = \ln t$ and $\mathrm{d}x = \dfrac{\mathrm{d}t}{t}$, and then

$$J(x) = \int \frac{1}{t\sqrt{1+2t}}\,\mathrm{d}t.$$

Let $u = \sqrt{1+2t}$, then $t = \dfrac{1}{2}(u^2-1)$ and $\mathrm{d}t = u\,\mathrm{d}u$. So

$$J(x) = \int \frac{2\,\mathrm{d}u}{u^2-1} = -\ln\left|\frac{u+1}{u-1}\right| + \text{cnst} = -\ln\left|\frac{\sqrt{1+2e^x}+1}{\sqrt{1+2e^x}-1}\right| + \text{cnst}.$$

(b) **Solving the differential equation : $e^{-x}y' - \dfrac{1}{1+2e^x}y = e^{-x}$.** It's a LDE that be written as

$$y' - \frac{e^x}{1+2e^x}y = 1. \qquad (1)$$

We look y of the form $y = uv$. Then $y' = u'v + v'u$ and (1) becomes

$$u'v + u\left[v' - \frac{e^x}{1+2e^x}v\right] = 1. \qquad (2)$$

Calculation of v : We choose v such that : $v' - \dfrac{e^x}{1+2e^x}v = 0$. It's a SVDE that be written as :

$$\frac{\mathrm{d}v}{v} = \frac{e^x}{1+2e^x}\,\mathrm{d}x.$$

By integration, we find $\ln v = \dfrac{1}{2}\ln(1+2e^x)$. Hence $v = \sqrt{1+2e^x}$.

Calculation of u : From (2) we get

$$u'v = 1 \Rightarrow u' = \frac{1}{\sqrt{1+2e^x}}.$$

By integration we find $u = J(x)$.

2. **Solving** $\boldsymbol{x\left(\tan\frac{y}{x}\right)^2 y' = x\left(\left(\cos\frac{y}{x}\right)^2+1\right)+y\left(\tan\frac{y}{x}\right)^2}$.

We have

$$y' = \frac{\left(\cos\frac{y}{x}\right)^2 + 1 + \frac{y}{x}\left(\tan\frac{y}{x}\right)^2}{\left(\tan\frac{y}{x}\right)^2}. \qquad (1)$$

It's a HDE. Let $z = \frac{y}{x}$, then $y' = xz' + z$ and (1) becomes :

$$xz' + z = \frac{\cos^2 z + 1 + z\tan^2 z}{\tan^2 z} \Rightarrow xz' = \frac{1+\cos^2 z}{\tan^2 z} \Rightarrow \frac{\tan^2 z}{1+\cos^2 z}\,\mathrm{d}z = \frac{\mathrm{d}x}{x}.$$

By integration, we find

- $\displaystyle\int \frac{\mathrm{d}x}{x} = \ln|x| + \text{cnst}.$
- $$\begin{aligned}\int \frac{\tan^2 z}{1+\cos^2 z}\,\mathrm{d}z &= \int \frac{\tan^2 z}{\frac{1}{\cos^2 z}+1}\times\frac{\mathrm{d}z}{\cos^2 z} = \int\frac{\tan^2 z}{2+\tan^2 z}\times\frac{\mathrm{d}z}{\cos^2 z}\\ &\underset{t=\tan z}{=} \int\frac{t^2}{2+t^2}\,\mathrm{d}t = \int\left(1-\frac{2}{2+t^2}\right)\mathrm{d}t\\ &= t-\sqrt{2}\arctan\frac{t}{\sqrt{2}} + \text{cnst}.\end{aligned}$$

Exercise 4

---

1. $$I(t) = \int \frac{\mathrm{d}\left(t-\frac{1}{2}\right)}{\frac{5}{4}-\left(t-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{5}}\ln\left|\frac{t-\frac{1}{2}+\frac{\sqrt{5}}{2}}{t-\frac{1}{2}-\frac{\sqrt{5}}{2}}\right| + \text{cnst}.$$

2. **Calculation of** $\boldsymbol{J(x) = \int x\frac{\cos x - 2\sin x}{(2\cos x+\sin x)^2}\,\mathrm{d}x}$**.** Integrate by parts with

$$\begin{cases} u = x \\ \mathrm{d}v = \dfrac{\cos x - 2\sin x}{(2\cos x+\sin x)^2}\,\mathrm{d}x\end{cases} \Rightarrow \begin{cases}\mathrm{d}u = \mathrm{d}x \\ v = -\dfrac{1}{2\cos x+\sin x}\end{cases}$$

we get then

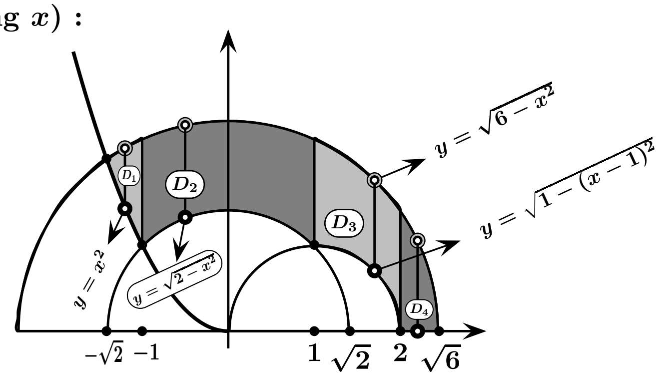
$$J(x) = uv - \int v\,\mathrm{d}u = -\frac{x}{2\cos x+\sin x} + \int\frac{\mathrm{d}x}{2\cos x+\sin x}.$$

Let's calculate $K = \int\frac{\mathrm{d}x}{2\cos x+\sin x}$. Let $t = \tan\frac{x}{2}$ then $\mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}$ with $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$. Hence :

$$K = \int\frac{\frac{2\,\mathrm{d}t}{1+t^2}}{2\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} = I(t).$$

Exercise 5

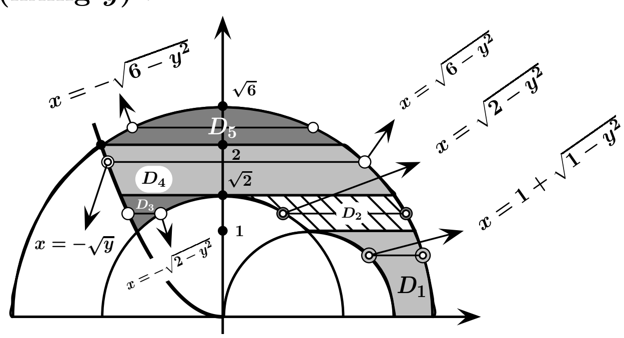
1° **Vertically (fixing x) :**



Then

$$I=\int_{-\sqrt{2}}^{-1}\mathrm{d}x\int_{x^2}^{\sqrt{6-x^2}}f(x,y)\,\mathrm{d}y+\int_{-1}^{1}\mathrm{d}x\int_{\sqrt{2-x^2}}^{\sqrt{6-x^2}}f(x,y)\,\mathrm{d}y+\int_{1}^{2}\mathrm{d}x\int_{\sqrt{1-(x-1)^2}}^{\sqrt{6-x^2}}f(x,y)\,\mathrm{d}y+\int_{2}^{\sqrt{6}}\mathrm{d}x\int_{0}^{\sqrt{6-x^2}}f(x,y)\,\mathrm{d}y.$$

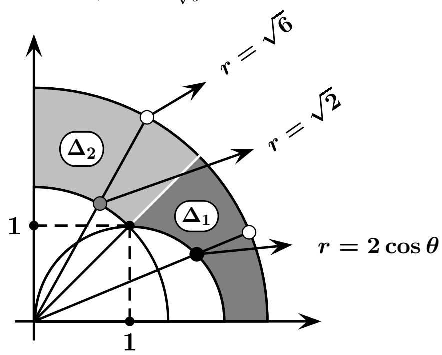
Horizontally (fixing y) :



$$I=\int_0^1\mathrm{d}y\int_{1+\sqrt{1-y^2}}^{\sqrt{6-y2}}f(x,y)\,\mathrm{d}x+\int_1^{\sqrt{2}}\mathrm{d}y\int_{-\sqrt{y}}^{-\sqrt{2-y^2}}f(x,y)\,\mathrm{d}x+\int_1^{\sqrt{2}}\mathrm{d}y\int_{\sqrt{2-y^2}}^{\sqrt{6-y^2}}f(x,y)\,\mathrm{d}x$$

$$+\int_{\sqrt{2}}^{2}\mathrm{d}y\int_{-\sqrt{y}}^{\sqrt{6-y^2}}f(x,y)\,\mathrm{d}x+\int_{2}^{\sqrt{6}}\mathrm{d}y\int_{-\sqrt{6-y^2}}^{\sqrt{6-y^2}}f(x,y)\,\mathrm{d}x.$$

2°



By polar coordinates :

$I=I_1+I_2$ where

$$I_1=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_{2\cos\theta}^{\sqrt{6}}f(r\cos\theta,r\sin\theta)r\,\mathrm{d}r$$

$$I_2=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_{\sqrt{2}}^{\sqrt{6}}f(r\cos\theta,r\sin\theta)r\,\mathrm{d}r$$